MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 98 No. 2

MEETING OF 1937 DECEMBER 10

Dr. H. Spencer Jones, President, in the Chair.

The election by the Council of the following Fellow was duly confirmed :—

Arthur P. Rollett, B.Sc., 4 Oak Lane, Sevenoaks, Kent.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

Joseph Bernard Ayscough-Clarke, M.C.I., M.I.St.E., M.I.E.E., Court Mansions, Penywern Road, London, S.W. 5 (proposed by J. J. Steward).

Eighty-one presents were announced as having been received since the last meeting, including :—

Halle Academy, "Festgabe," for the celebration of the 250th anniversary of the Kaiserlich Leopoldinisch Akademie.

Mr. E. A. Chamberlain and Mr. W. E. Candler were appointed Auditors of the Treasurer's accounts for the year 1937.

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THE SYSTEM OF α URSÆ MAJORIS.

By H. Spencer Jones, F.R.S., and H. H. Furner.

The duplicity of the star, α Ursæ Majoris, was discovered by Burnham in 1889, with the 36-inch refractor of the Lick Observatory. The separation was about $0''.9$, and the companion was estimated to be of about the 11th magnitude. In succeeding years further observations by Burnham and others showed that there was a rapid orbital motion. By 1902 the pair had become extremely difficult to measure, the separation having decreased to about $0''.6$. For many years the star was no longer seen double. Though duplicity was suspected by Aitken in 1925, it was not until 1933 that the companion was seen with certainty. In that year the separation and position angle did not differ much from their values at Burnham's first observation in 1889, showing that the period was approximately 44 years. At the present time the pair is not a difficult one to measure, and during recent years measures have been made by Aitken, with the Lick 36-inch refractor; by Baize, with the Paris 31-cm. refractor; and by Woolley and Symms, with the Greenwich 28-inch refractor.

The estimates of the brightness of the companion have varied widely. Burnham assigned a magnitude of $11^m.1$; Aitken estimated it to be $9^m.0$. More recently Kuiper,* by photometric measurements, obtained a magnitude of $4^m.89$; whilst Baize † estimates the magnitude to be about $6^m.5$. The wide variation in these estimates emphasises the difficulty of assigning the correct magnitude to a faint star in close proximity to one that is very much brighter.

As the range of position angles within which reliable observations have been made is less than 60°, it is impossible from the double-star observations to derive reliable orbital elements. The uncertainty in the orbital elements is indicated by the two provisional sets of elements that have been published. Aitken ‡ estimated the eccentricity to have a moderate value, 0·25 to 0·30, whereas Baize § has given a value of 0·74. The one element that can be fixed with some accuracy from the double-star observations is the period. If, from other data, the eccentricity and the time of periastron can be assigned, the double-star observations will serve to give a good determination of the remaining elements.

It was announced by Campbell, on the basis of spectrograms secured in the years 1897 to 1909 at the Lick Observatory, that the radial velocity was variable. The range in velocity is, however, so small that the radial velocity

* *P.A.S.P.*, 47, 25, 1935.
† *L'Astronomie*, 60, 384, 1936.
‡ *P.A.S.P.*, 45, 190, 1933.
§ *L'Astronomie*, 60, 384, 1936.

measures do not materially help, in the absence of further information, to give precision to the orbital elements.

A preliminary examination of the meridian observations of α Ursæ Majoris made at Greenwich gave some indication of variable proper motion. It appeared possible that, if all the available meridian observations were collected, the additional material so provided, used in conjunction with the double-star and radial velocity measures, would enable reasonably accurate orbital elements to be determined.

The results of this investigation are contained in the present paper. The collection of the data of the meridian observations and their analysis were completed early in 1934. Some difficulties were found in harmonising the radial velocity data, available at that time, with the other data. Dr. Aitken had reported that there was a suspicion of a short-period variation in the radial velocities, and that the matter was being investigated by Dr. Moore. The completion of the discussion of the various data was consequently deferred until this investigation had been made. A summary of the further radial velocity measures made at the Lick Observatory was recently received from Dr. Moore. This additional material has been of considerable help, and we much appreciate Dr. Moore's kindness in forwarding the results in advance of publication and allowing us to use them.

I. *The Meridian Observations*

The extreme range of variation, produced by orbital motion, in both the right ascension and the declination of the bright component is of the order of 0″·4. In order to determine these small variations with as much accuracy as possible, the method adopted by Bessel* for the investigation of the variable proper-motion of Sirius was used. In this method the right ascension and declination of the star whose variable proper-motion is to be investigated are compared with the mean right ascension and declination of a certain number of suitably selected other stars. The systematic errors, with which the meridian observations are affected, are to a large extent eliminated by this differential method.

The right ascensions of α Ursæ Majoris were compared with the mean right ascensions of the following four stars : α Aurigæ ; γ Ursæ Majoris ; η Ursæ Majoris and β Ursæ Minoris. The declinations were compared with the mean declinations of the following six stars : α Cassiopeiæ ; α Persei ; γ Ursæ Majoris ; β Ursæ Minoris ; α Cephei and β Cephei. For each observatory the mean position for a year or for a group of years of each of the above stars was compared with the position for the corresponding epoch, computed from the *Preliminary General Catalogue* of Boss. The differences between the observed and tabular positions were formed. From the differences for α Ursæ Majoris the mean of the corresponding differences for the companion stars was subtracted. The residuals so obtained are denoted by q for right ascensions, and r for declinations. The quantities

* *A.N.*, Nos. 745–8, 1851.

q and r are free from errors in the adopted equinox and latitude, and are comparatively free from systematic errors that vary with right ascension and declination, as well as from errors arising from the use of incorrect refractions. They contain a constant part, depending upon the errors of the tabular places at the mean epoch covered by the observations ; a part varying linearly with time, resulting from errors in the tabular proper motions of Boss's Catalogue ; and a periodic part, due to the variable proper-motion of α Ursæ Majoris, resulting from the orbital motion.

Corrections were applied, where necessary, to reduce the observations to the value, 9″·210, of the constant of nutation. In many of the earlier catalogues either Bessel's value (9″·250) or Peters's value (9″·224) were used. This correction is important because nutation is a long-period term. Errors introduced through the use of erroneous values of the constant of aberration can be neglected because their maximum effect is not large, and in the mean of a number of observations the errors become accidental and are not of a systematic nature.

The values of q and r used in the subsequent discussion are collected in Tables I and II respectively. The first column gives the name of the observatory, the second the period covered by the observations, the third the mean epoch of the observation, and the fourth the assigned weight. The weights are necessarily to some extent arbitrary, but an attempt has been made to assign weights such that the probable error corresponding to unit weight is $\pm$ 0″·30.

The sum of the residuals in right ascension showed that a correction to the mean proper-motion in right ascension was needed. A correction of +0^s·000353 (t-1850) was applied to these residuals. The residuals in declination did not require the application of a proper-motion correction.

TABLE I

Values of q

Observatory	Period of Observation	Mean Epoch	Observed q	Weight
			s	
Greenwich	1811–19	1817·6	+·050	1·0
Dorpat	1818–22	21·3	·000	1·5
Königsberg	1822–26	24·5	−·042	2·0
Åbo	1827–30	29·0	+·005	4·0
Greenwich	1828–30	30·0	+·103	0·3
,,	1831–32	32·1	+·066	0·4
Madras	1831–42	32·8	+·020	0·5
Greenwich	1833	33·5	+·024	0·6
,,	1834–35	34·7	−·031	1·0
Edinburgh	1834–45	36·9	+·010	1·5
Greenwich	1836–37	37·1	+·056	1·0
Paris	1837	37·6	+·007	2·5
,,	1838	38·7	−·003	2·5
Greenwich	1838–40	39·0	+·070	1·0

TABLE I—*continued*

Observatory	Period of Observation	Mean Epoch	Observed q	Weight
			s	
Paris	1839	39·5	+·010	2·0
,,	1840	40·6	−·036	2·0
,,	1841	41·6	−·064	1·5
Greenwich	1841–42	42·0	+·169	1·0
Geneva	1841–42	42·0	−·099	1·0
Radcliffe	1840–53	42·6	+·132	0·8
Paris	1842	42·6	−·067	1·5
Madras	1843	43·5	−·080	0·7
Paris	1843	43·6	−·038	1·5
Greenwich	1843–44	44·0	+·167	1·0
Geneva	1843–45	44·4	+·003	1·0
Madras	1844	44·5	+·098	1·5
Paris	1844–45	45·0	+·003	1·5
Pulkova	1842–49	45·5	+·032	6·0
Madras	1845	45·5	+·105	1·0
Washington	1845–46	46·2	+·021	2·0
Pulkova	1842–49	46·4	+·047	4·0
Greenwich	1845–47	46·4	−·030	1·5
Madras	1846	46·5	+·258	0·7
Paris	1846	46·7	−·012	1·5
Madras	1847	47·5	−·189	0·7
Paris	1847	47·5	−·053	2·0
Geneva	1847–48	48·0	−·027	1·0
Madras	1848	48·5	−·064	0·7
Washington	1847–48	48·8	+·145	2·0
Brussels	1848–49	49·0	−·070	1·0
Paris	1848–49	49·0	−·023	2·0
Greenwich	1848–49	49·1	+·074	2·5
Madras	1849	49·5	−·080	0·7
Geneva	1849–50	50·0	−·039	1·0
Madras	1850	50·5	+·045	0·7
Greenwich	1850–53	1851·2	−·030	1·5
Paris	1850–53	51·5	+·035	2·0
Washington	1851–52	51·9	+·081	1·0
Geneva	1851–53	52·3	−·033	1·0
Madras	1852	52·5	+·038	0·8
Greenwich	1854–55	55·0	+·062	3·0
Geneva	1854–56	55·5	+·007	1·0
Paris	1854–56	56·0	+·036	2·0
Radcliffe	1854–61	56·2	+·132	1·0
Brussels	1855–56	56·2	−·053	1·0
Greenwich	1856–58	58·0	−·033	2·5
Geneva	1857–58	58·0	+·001	1·0
Paris	1857–59	58·5	+·033	2·0
Brussels	1857–63	58·9	+·115	1·0
Washington	1859–60	59·8	−·037	1·0
Greenwich	1859–60	60·2	−·021	2·0
Washington	1861–63	62·8	+·042	1·5
Greenwich	1861–65	63·2	−·039	2·0
Paris	1860–68	63·6	+·107	1·5

TABLE I—*continued*

Observatory	Period of Observation	Mean Epoch	Observed q s	Weight
Pulkova	1861–71	64·9	+·026	7·0
Bonn	1865–67	66·3	+·025	1·0
Greenwich	1866–67	67·2	−·021	4·0
Washington	1865–69	67·5	−·038	3·0
Leipzig	1866–67	67·6	+·092	1·0
Radcliffe	1864–70	68·6	−·003	2·0
Greenwich	1868–70	69·3	−·096	6·0
Brussels	1864–74	69·5	+·030	1·0
Berlin	1869–73	72·0	−·025	6·0
Washington	1870–73	72·0	−·042	3·0
Harvard	1871–72	72·3	−·069	1·0
Greenwich	1871–73	72·9	+·003	4·0
Harvard	1870–79	74·3	−·019	5·0
Washington	1874–75	75·0	−·005	3·0
Greenwich	1874–76	75·5	−·013	3·0
Radcliffe	1871–76	76·6	+·041	2·0
Pulkova	1874–80	77·4	−·058	3·0
Greenwich	1877–80	79·9	−·043	4·0
,,	1881–86	84·5	−·015	4·0
Pulkova	1882–91	84·5	+·018	8·0
Radcliffe	1880–94	86·2	+·162	0·5
Pulkova	1881–94	87·2	−·065	3·0
Lisbon	1887–90	90·4	−·013	0·5
Greenwich	1892–94	93·6	−·035	5·0
Berlin	1892–97	94·1	+·002	2·0
Greenwich	1895–96	95·9	+·040	6·0
,,	1897–99	98·6	+·054	4·0
Pulkova	1899–03	1901·2	+·032	6·0
Greenwich	1900–02	01·4	+·058	3·0
,, (Altaz.)	1899–08	01·8	−·055	1·0
Kiev	1900–05	1903·8	−·045	1·0
Greenwich	1903–05	03·9	+·032	2·0
Munich	1901–07	04·9	−·076	5·0
Pulkova	1903–06	05·9	+·010	7·0
Washington	1903–11	06·7	−·117	3·0
Lick	1907–08	08·5	+·051	1·0
Greenwich	1906–14	11·1	−·009	3·0
Paris	1909–14	11·7	−·044	2·0
Pulkova	1911–18	13·0	−·015	10·0
Washington	1911–18	13·7	−·098	4·0
Albany	1911–18	14·3	−·045	4·0
Algiers	1916–17	17·0	−·058	3·0
Washington	1913–26	18·4	−·054	1·0
Greenwich	1915–21	19·2	−·014	5·0
,,	1922–24	23·4	−·057	5·0
Berlin	1922–23	23·4	−·026	2·0
Greenwich	1925–27	26·8	−·001	7·0
Berlin	1927–30	29·0	−·012	1·0
Greenwich	1928–30	29·5	−·015	6·0
,,	1931–33	32·9	+·008	8·0

TABLE II

Values of r

Observatory	Period of Observation	Mean Epoch	Observed r	Weight
			″	
Groombridge	1806–16	1808·7	−0·18	0·6
Greenwich	1813	12·5	− ·52	0·4
,,	1811–19	14·7	− ·35	3·0
,,	1820–21	21·2	+ ·35	1·0
,,	1822–23	23·1	+ ·69	1·0
,,	1824	24·5	+ ·72	0·8
Königsberg	1822–26	24·5	− ·27	2·5
Greenwich	1825–27	26·5	+ ·13	3·0
Åbo	1827–30	29·0	+ ·09	4·0
Greenwich	1828–30	30·2	− ·32	1·0
,,	1831–32	32·1	− ·41	1·0
Madras	1831–42	33·0	− ·09	0·8
Greenwich	1833–34	34·0	− ·30	1·0
Cambridge	1828–35	34·0	− ·32	2·0
Greenwich	1835–36	36·2	− ·54	2·5
Cambridge	1836–38	37·7	− ·21	1·5
Greenwich	1837–38	38·1	− ·21	2·5
Königsberg	1836–40	38·5	+ ·54	2·0
Edinburgh	1834–45	38·9	− ·56	2·5
Greenwich	1839–41	40·5	− ·61	2·5
Cambridge	1839–42	1841·2	− ·32	1·5
Paris	1837–44	41·4	+ ·02	3·0
Pulkova	1842–49	42·6	− ·21	5·0
Geneva	1841–45	43·4	+ ·70	1·0
Greenwich	1842–44	43·7	− ·32	3·0
Madras	1843–44	44·1	+ ·06	1·0
Cambridge	1843–48	45·0	− ·40	1·5
Madras	1845–46	46·0	+ ·25	1·0
Greenwich	1845–47	46·3	− ·35	3·0
Radcliffe	1840–53	46·5	+ ·58	0·7
Washington	1845–48	46·7	− ·30	1·5
Madras	1847–48	48·0	+ ·43	0·7
Paris	1845–51	48·1	− ·11	3·0
Geneva	1847–49	48·8	·00	0·7
Greenwich	1848–51	49·1	− ·06	3·0
Madras	1849–52	50·7	+ ·42	1·0
Geneva	1850–54	52·1	− ·46	0·7
Paris	1851–54	52·8	+ ·01	1·0
Greenwich	1852–54	53·4	+ ·02	2·5
Paris	1852–55	53·5	− ·01	3·0
Washington	1851–62	54·1	+ ·60	0·7
Radcliffe	1854–61	56·6	+ ·92	0·3
Brussels	1855–57	56·8	+ ·43	0·5
Geneva	1855–58	57·0	− ·71	0·7
Greenwich	1856–58	57·5	− ·21	3·0
Paris	1855–60	57·4	− ·06	4·0
Brussels	1859–66	63·0	− ·60	0·5
Paris	1859–66	63·5	− ·17	2·5

TABLE II—*continued*

Observatory	Period of Observation	Mean Epoch	Observed r	Weight
			″	
Greenwich	1861–67	65·3	+ ·11	4·0
Leiden	1863–68	65·6	+ ·24	6·0
Bonn	1865–67	66·3	+ ·23	1·0
Pulkova	1861–71	66·6	+ ·55	8·0
Washington	1863–69	66·7	+ ·35	3·5
Radcliffe	1864–68	67·1	+ ·69	0·6
Leipzig	1866–67	67·6	+ ·07	2·0
Greenwich	1868–70	69·4	− ·01	3·5
Radcliffe	1869–72	70·8	+ ·79	0·8
Washington	1870–75	73·1	+ ·33	5·0
Harvard	1871–75	73·7	− ·28	2·0
Greenwich	1871–76	74·1	− ·31	4·0
Radcliffe	1873–76	75·2	− ·36	0·8
Pulkova	1874–80	77·4	− ·08	5·0
Washington	1876–79	77·8	− ·08	5·0
Greenwich	1877–82	80·4	− ·06	6·0
Radcliffe	1880–84	81·3	− ·25	3·0
Washington	1880–84	82·3	+ ·08	6·0
Greenwich	1883–86	84·9	− ·16	5·0
Pulkova	1881–94	85·6	− ·05	9·0
Washington	1885–87	1886·4	+ 0·12	3·0
Greenwich	1887–94	92·5	− ·36	6·0
Berlin	1892–97	94·1	·00	2·0
Greenwich	1895–96	96·0	− ·43	6·0
,,	1897–00	99·8	− ·09	7·0
Odessa	1901–02	1901·1	+ ·32	8·0
Washington	1898–07	03·1	+ ·10	2·5
Greenwich	1901–05	03·1	+ ·09	5·0
Kiew	1900–05	03·8	+ ·17	2·0
München	1901–07	04·8	− ·17	3·0
Pulkova	1903–06	05·4	+ ·12	5·0
Washington	1903–11	06·7	+ ·23	3·5
Lick	1907–08	08·0	− ·27	1·0
Greenwich	1906–15	11·2	+ ·11	6·0
Paris	1909–14	11·9	+ ·25	2·0
Washington	1911–18	13·6	− ·21	5·0
Pulkova	1911–18	14·0	+ ·19	7·0
Albany	1911–18	14·3	+ ·21	3·0
Algiers	1913–21	17·0	+ ·20	3·0
Washington	1913–26	18·4	+ ·30	2·0
Greenwich	1915–21	19·3	− ·14	5·0
Berlin	1916–27	21·9	− ·49	1·0
Bonn	1921–27	23·3	+ ·01	1·0
Poznan	1924–27	26·0	− ·17	1·0
Greenwich	1922–30	27·8	− ·19	8·0
Berlin	1927–30	29·0	− ·93	1·0
,,	1928–30	31·5	− ·12	1·0
Greenwich	1931–33	32·5	− ·33	8·0

When the values of q and r, given in Tables I and II, were measured at intervals of a few years, a periodic error in the residuals was clearly shown, with a period of about 43 or 44 years. The period can be more accurately determined from the double-star observations and has been assumed to be 44·0 years. This period is not likely to be more than 6 months in error.

Using this period, the observations were reduced to one revolution, and 12 normal places were formed in both right ascension and declination. These normal places are given in Tables III and IV. The phases are measured from 1864·0. Constant corrections have been applied, of $-0^s\cdot0050$ to the values of q, and of $-0''\cdot030$ to the values of r, to give an approximate adjustment of zero to the mean values.

TABLE III

Normal Places in Right Ascension

(Phases measured from 1864·0)

Phase in Years	Mean Observed q (Time)	Mean Observed q (Arc)	Weight	Computed q	Δq (O − C)
y	s	"		"	"
1·72	−0·0064	−0·044	28·0	−0·119	+0·075
5·16	− ·0310	− ·217	31·0	− ·198	− ·019
8·80	− ·0199	− ·139	19·0	− ·147	+ ·008
10·28	− ·0066	− ·046	14·8	− ·117	+ ·071
13·10	− ·0160	− ·112	15·0	− ·061	− ·051
17·56	+ ·0005	+ ·003	25·5	+ ·014	− ·011
21·65	− ·0015	− ·010	27·0	+ ·068	− ·078
25·33	+ ·0189	+ ·132	38·4	+ ·108	+ ·024
29·10	+ ·0140	+ ·098	11·6	+ ·140	− ·042
32·74	+ ·0285	+ ·199	31·8	+ ·160	+ ·039
37·90	+ ·0380	+ ·266	13·0	+ ·151	+ ·115
40·84	·0000	·000	26·5	+ ·099	− ·099

TABLE IV

Normal Places in Declination

Phase in Years	Mean Observed r	Weight	Computed r	Δr (O − C)
y	"		"	"
1·39	+0·084	25·5	+0·171	−0·087
3·39	+ ·328	20·4	+ ·098	+ ·230
5·90	+ ·030	15·0	− ·020	+ ·050
8·31	+ ·025	22·3	− ·103	+ ·128
11·20	− ·133	16·8	− ·158	+ ·025
14·50	− ·120	16·8	− ·184	+ ·064
18·68	− ·183	35·0	− ·186	+ ·003
21·75	− ·160	24·0	− ·174	+ ·014
24·71	− ·266	22·7	− ·155	− ·111
30·20	− ·209	26·6	− ·098	− ·111
36·36	+ ·024	27·2	+ ·010	+ ·014
39·64	− ·045	21·5	+ ·095	− ·140

The periodic error in both q and r is clearly shown in Tables III and IV. The variations point to a moderate value of the eccentricity, but it is evident that the scatter in the observations is not inappreciable in comparison with the total ranges of variation of q and r, and that the eccentricity cannot be determined with very high precision. The last two columns of Tables III and IV contain the values of q and r computed from the final orbital elements, given later in this paper, and the residuals between the observed and computed values.

II. *The Double Star Observations*

The double-star observations have been collected in Table V, in which observations by two observers have in some cases been combined, so as to give mean positions of approximately equal weight. The measured position angles and distances have been used to give values of $\Delta\alpha$, $\Delta\delta$, for analysis in a manner similar to that used for the meridian observations.

TABLE V

Double Star Observations

Date	Observer	Observed		$\Delta\alpha$	$\Delta\delta$	Computed		Residuals (O − C)	
		θ	ρ			θ	ρ	$\Delta\theta$	$\Delta\rho$
		°	″	″	″	°	″	°	″
1889·19	β4	326·1	0·91	+0·51	−0·75	321·7	0·85	+4·4	+0·06
90·26	β4	320·1	·87	+ ·56	− ·67	318·7	·85	+1·4	+ ·02
90·53	HΣ3	316·7	·81	+ ·56	− ·59	318·1	·85	−1·4	− ·04
91·30	β4	316·8	·80	+ ·55	− ·58	315·8	·84	+1·0	− ·04
92·80	β3 Bar. 2	310·2	·90	+ ·69	− ·58	311·4	·83	−1·2	+ ·07
95·78	Com. 4	303·4	·84	+ ·70	− ·46	301·6	·81	+1·8	+ ·03
98·16	β2 Com. 2	293·4	·79	+ ·73	− ·31	292·4	·77	+1·0	+ ·02
99·31	Com. 3 A1	281·2	·76	+ ·75	− ·15	287·1	·74	−5·9	+ ·02
1902·88	Com. 1 A1	272·6	·55±	+ ·55	− ·02	264·6	·64	+8·0	− ·09
33·51	A3	322·7	·85	+ ·52	− ·68	320·9	·85	+1·8	·00
35·33	A3	315·4	·82	+ ·58	− ·58	315·7	·84	−0·3	− ·02
36·37	Baize 6	312·2	·86	+ ·64	− ·58	312·8	·84	−0·6	+ ·02
37·27	Gwh. 7	308·2	·80	+ ·63	− ·49	310·0	·83	−1·8	− ·03

Aitken's observation in 1902 is marked "uncertain." He adds that "the pair was extremely difficult to measure in 1902 and the result given is correspondingly uncertain." *

It is stated by Aitken that he saw no companion in 1905·266, 1911·336, and at other dates between. Comstock apparently did not examine the star between 1903 and 1908. In the latter year the companion was not seen, nor in later years, though it was looked for on many nights. Aitken reports in 1933 : "I examined the star repeatedly from 1911 up to the present year, using the 36–inch telescope under good observing conditions, but without success. In the early spring of 1933 the weather was unfavourable ; but when I turned to the star on the evening of July 5 the companion was

* *Lick Obs.*, 12, 69, 1914.

seen distinctly, and measures were made without special difficulty." * With the 28–inch refractor at Greenwich, the companion could not be seen in 1937, when the full aperture was used, but, on changing the diffraction pattern by considerably stopping down the objective, the companion became visible and measures could be made without any great difficulty.

The double-star observations cannot be used by themselves to determine the orbit. If the eccentricity and time of periastron are known, however, they enable the other elements of the orbit to be derived more accurately than they can be derived from the meridian observations.

The computed values of θ and ρ in Table V are based upon the final orbital elements, given later.

III. *Discussion of the Meridian and Double Star Observations*

The observations are most conveniently discussed by the method originally used by Bessel for the investigation of the variable proper motion of Sirius.

The components of the orbital displacement at any time can be expressed in the form :

$$\Delta\alpha = -ge = +g \cos u + h \sin u,$$
$$\Delta\delta = -g'e \quad +g' \cos u + h' \sin u,$$

where $g = B$; $h = (1 - e^2)^{\frac{1}{2}}G$; $g' = A$; $h' = (1 - e^2)^{\frac{1}{2}}F$; u denotes the eccentric anomaly, e is the eccentricity, and $\Delta\alpha$ is expressed in arc.

A, B, F, G are identical with the so-called Thiele-Innes constants, and, as Innes's designations for these are now extensively used in double-star literature, they are employed here. They are defined by

$$A = a(\cos\omega \cos\Omega - \sin\omega \sin\Omega \cos i),$$
$$B = a(\cos\omega \sin\Omega + \sin\omega \cos\Omega \cos i),$$
$$F = a(-\sin\omega \cos\Omega - \cos\omega \sin\Omega \cos i),$$
$$G = a(-\sin\omega \sin\Omega + \cos\omega \cos\Omega \cos i),$$

ω denotes the angle between the line of nodes and the major axis, Ω is the position angle of the line of nodes, and i is the inclination of the orbit plane, which is to be taken between 90° and 180° in the case of retrograde motion, and a is the semi-axis major of the orbit.

When dealing with the meridian observations, the terms $-ge$, $-g'e$ in the expressions for $\Delta\alpha$, $\Delta\delta$ are to be replaced by constants c, c', because the position of the zero point from which the displacements are measured is arbitrary.

For the application of these formulæ it is necessary first to determine approximately correct values of the period, the eccentricity and the time of periastron. The value of u for each observation is then known and a least squares solution will give the corresponding values of g, h, g', h' and therefore of a, i, ω and Ω. If the initial values of the period, eccentricity and time of periastron are not greatly in error, differential corrections may be determined, provided that the observational material justifies the labour.

* *P.A.S.P.*, 45, 190, 1935.

A period of 44·0 years has already been assumed. Some tentative solutions led to the adoption of $T_0 = 1865{\cdot}9$, $e = 0{\cdot}35$.

Equations of condition were formed from the normal places, derived from the meridian observations, given in Tables III and IV. Using the weights given in these tables, normal equations were formed and solved for g, h, g', h'. From these the values of the constants A, B, F, G were deduced.

Equations of condition were also formed for each double-star position given in Table V. Equal weight was given to each observation, normal equations were formed and the corresponding values of A, B, F, G were derived.

The results of these two solutions were as follows :—

	Meridian Observations	Double Star Observations
	″ ″	″ ″
A	+0·197 ± 0·027	+0·519 ± 0·010
B	− ·100 ± ·020	− ·364 ± ·010
F	+ ·002 ± ·028	− ·320 ± ·027
G	− ·161 ± ·021	− ·480 ± ·026

There should be a constant ratio between the values of the four quantities derived from the meridian and double-star observations respectively, as the quantities determine similar orbits, viz. the orbit of the bright star about the centre of gravity and the orbit of the faint star relative to the bright.

It will be noted that the agreement is satisfactorily good for A, B and G, but that the values of F are entirely discordant. This discordance persists for any reasonable changes in the eccentricity and time of periastron. It indicates that a smooth curve representing the displacements in declination given in Table IV differs in shape from that which is required to fit the double-star observations. This difference in shape is probably to be attributed to a fortuitous accumulation of errors in the meridian observations.

Though the double-star observations are insufficient in themselves to determine the orbital elements, the above results show that when T_0 and e have been fixed, the remaining elements can be derived with much greater accuracy from the double-star observations than from the meridian observations.

It may be remarked that the probable error for unit weight derived from the right ascension observations was ± 0″·22, and that derived from the declination observations was ± 0″·30. As mentioned above, an attempt was made, when assigning the weights, to fix them on the basis of a probable error for unit weight of ± 0″·30. It appears that the weights in right ascension were somewhat underestimated. The probable error in either co-ordinate of each double-star position is about ± 0″·027.

The two series of normal equations, derived from the meridian and double-star observations respectively, were therefore combined with multiplying factors based on the relative probable errors. The constant terms in the double-star equations were multiplied by a factor k, representing the ratio of the semi-axis major of the meridian orbit to that of the double-

star orbit. It is obvious that k determines the ratio of the mass of the faint star to the combined mass.

The combined normal equations were solved and A, B, F, G were derived in terms of k as an undetermined factor. The following results were obtained:—

$$\begin{aligned} A &= +0''.0038 + 0''.504\, k, \\ B &= +0''.0036 - 0''.379\, k, \\ F &= -0''.0073 - 0''.278\, k, \\ G &= -0''.0205 - 0''.405\, k. \end{aligned}$$

The sums of the squares of the residuals were formed. In right ascension the sum was a minimum for $k = \cdot 303$; in declination, the sum was a minimum for $k = \cdot 315$. The mean value, $k = \cdot 309$, was adopted.

With this value we obtain :

$$\begin{aligned} A &= +\cdot 1595'' \pm \cdot 0034'', \\ B &= -\cdot 1135'' \pm \cdot 0033'', \\ F &= -\cdot 0932'' \pm \cdot 0089'', \\ G &= -\cdot 1456'' \pm \cdot 0085''. \end{aligned}$$

Using

$$\tan(\omega - \Omega) = \frac{-B - F}{A - G}, \qquad \tan(\omega + \Omega) = \frac{B - F}{A + G}$$

we have

$$\begin{aligned} \omega - \Omega &= 34^\circ.1 \pm 1^\circ.5, \\ \omega + \Omega &= 304^\circ.4 \pm 20^\circ.8. \end{aligned}$$

The value of $(\omega + \Omega)$ is weakly determined, because $(B - F)$ and $(A + G)$ are both small. The determination of ω and Ω can be appreciably strengthened, however, by the use of the radial velocity measurements.

IV. *The Radial Velocity Measurements*

The radial velocity measures made at the Lick Observatory are summarised in Table VI. The earlier results have been taken from *Lick Obs. Pub.* **16** ; the results from 1927 onwards were communicated by Dr. J. H. Moore. In view of the possible existence of a short-period variation, the range in the measured velocities, when two or more plates have been combined to form a mean value, is given in the last column.

The range in velocity arising from the orbital motion with a period of 44 years is evidently small, and it is not possible from the observations to derive complete orbital elements. If we represent the radial velocity in the form :

$$\begin{aligned} v &= G + K\cos(u + \omega) \\ &= G + K_1 \cos u + K_2 \sin u, \end{aligned}$$

then, assuming the time of periastron and the eccentricity already adopted, the values of K_1 and K_2 can be determined by a least square solution, using the weights given in column 3 of Table VI.

The resulting values are :

$$K_1 = K \cos \omega = +2{\cdot}008 \pm 0{\cdot}21 \text{ km./sec.}$$
$$K_2 = -K \sin \omega = -0{\cdot}027 \pm 0{\cdot}28 \text{ km./sec.}$$
$$G = -\ 8{\cdot}72 \text{ km./sec.}$$

The probable error corresponding to unit weight (single plate) is $\pm$ 0·96 km./sec.

From the values of K_1 and K_2 we obtain

$$K = 2{\cdot}008 \pm 0{\cdot}21 \text{ km./sec.}$$
$$\omega = +0^\circ{\cdot}8 \pm 8^\circ{\cdot}0.$$

The angle ω is here assumed to be measured from the ascending node, at which the star is moving away from the observer. In the analysis of the meridian and double-star orbits, it has been measured from the other node. Hence we must take $\omega = 180^\circ{\cdot}8$ for combination with the previous results.

TABLE VI

Radial Velocity Measures of α Ursæ Majoris

Mean Date	Mean Observed Radial Velocity km./sec.	Weight	Range, km./sec.	Computed Radial Velocity km./sec.	ΔV (O − C) km./sec.
1897·23	− 9·8	5	− 9·4 to − 10·3	− 10·2	+0·4
97·98	− 9·6	1	...	− 10·1	+0·5
99·96	− 9·9	1	...	− 9·7	−0·2
1901·29	− 9·6	1	...	− 9·4	−0·2
03·99	− 9·2	1	...	− 8·6	−0·6
05·79	− 7·9	1	...	− 7·9	0·0
09·97	− 4·9	2	− 3·8 to − 6·0	− 6·7	+1·8
10·79	− 8·8	1	...	− 6·8	−2·0
11·42	− 7·1	3	− 5·8 to − 7·7	− 6·9	−0·2
17·10	− 9·8	2	− 9·7 ,, − 9·9	− 9·1	−0·7
22·18	− 10·2	4	− 7·8 ,, − 12·4	− 10·2	0·0
27·70	− 11·0	0·5	...	− 10·6	−0·4
33·53	− 10·6	2	− 10·2 to − 10·9 *	− 10·7	+0·1
34·26	− 10·8	6	− 9·5 ,, − 11·8	− 10·7	−0·1
34·86	− 9·4	1	...	− 10·7	+1·3
35·18	− 9·6	14	− 8·9 to − 10·4	− 10·7	+1·1
36·29	− 10·6	10	− 9·6 ,, − 11·8	− 10·6	0·0
36·91	− 10·1	6	− 9·7 ,, − 10·3	− 10·6	+0·5
37·37	− 10·8	78	− 9·2 ,, − 12·0	− 10·6	−0·2

* A discordant observation on July 14 (− 5·7 km./sec.) has been omitted.

V. *Combination of Results*

We have obtained the three following results :—

$$\omega - \Omega = 34^\circ{\cdot}1 \pm 1^\circ{\cdot}5,$$
$$\omega + \Omega = 304{\cdot}4 \pm 20{\cdot}8,$$
$$\omega = 180{\cdot}8 \pm 8{\cdot}0.$$

From these we obtain the adjusted values :

$$\omega = 174°{\cdot}0 \pm 8°{\cdot}3,$$
$$\Omega = 139°{\cdot}2 \pm 9°{\cdot}0,$$

which we adopt as definitive values. The difference between ω and Ω is determined more accurately than either quantity separately. It may be noted that these values satisfy the three equations above within the assigned probable errors.

With these values of ω and Ω and the values of A, B, F, G derived in § III above, we can obtain a and $a \cos i$ from the relations :

$$a = A \cos\omega \cos\Omega + B \cos\omega \sin\Omega - F \sin\omega \cos\Omega - G \sin\omega \sin\Omega,$$
$$a \cos i = -A \sin\omega \sin\Omega + B \sin\omega \cos\Omega - F \cos\omega \sin\Omega + G \cos\omega \cos\Omega.$$

These expressions give

$$a = +0''{\cdot}1958 \pm 0''{\cdot}0034,$$
$$a \cos i = -0''{\cdot}1702 \pm 0''{\cdot}0086,$$

whence

$$i = 150°{\cdot}3 \pm 5°{\cdot}5.$$

The semi-axis major of the relative orbit is obtained by dividing the above value of a by k, the value of which was found to be 0·309. We find that

$$a + a' = 0''{\cdot}634.$$

VI. *The Parallax and Mass of α Ursæ Majoris*

From the value of K deduced from the radial velocity observations, $a \sin i$, expressed in km., is derived from the formula

$$a \sin i = [6{\cdot}7009] \times KP(1 - e^2)^{-\frac{1}{2}},$$

where the period P is assumed to be expressed in years.

With $K = 2{\cdot}008$ km./sec., $P = 44$ years, we obtain

$$a \sin i = 4{\cdot}157 \times 10^8[1 \pm {\cdot}108] \text{ kms.}$$

Also

$$\sin i = 0{\cdot}495[1 \pm {\cdot}169],$$

whence

$$a = 8{\cdot}398 \times 10^8[1 \pm {\cdot}200] \text{ kms.}$$

Since a expressed in arc has the value $0''{\cdot}1958$, the parallax of the system can be inferred. The value obtained for the parallax is

$$\pi = 0''{\cdot}035 \pm 0''{\cdot}007.$$

The following values of the parallax have been given :—

Trigonometric	Greenwich	$+0''{\cdot}027 \pm 0''{\cdot}009$
	Mt. Wilson	$+0''{\cdot}018 \pm 0''{\cdot}008$
Spectroscopic	Mt. Wilson	$0''{\cdot}048$
	D.A.O.	$0''{\cdot}074$
Dynamical	Russell	$0''{\cdot}026$

The value that we have derived is in reasonably good accordance with the trigonometrical and dynamical determinations. The spectroscopic values appear to be too high.

The total mass of the system, in terms of the mass of the Sun as unit, can be derived from

$$m+m' = \frac{(a+a')^3}{\pi^3 P^2}.$$

This gives

$$m+m' = 3{\cdot}10.$$

The mass is not determined with very great accuracy. If the parallax were 0″·028, the combined mass would be 6·08, or nearly double the combined mass corresponding to a parallax of 0″·035.

A combined mass of 3·10 corresponds to a mass of 2·14 for the bright star and a mass of 0·96 for the faint star. Such a mass for the bright star is in serious disagreement with the mass to be expected according to the mass-luminosity relationship. It is of interest to examine what adjustment of elements would bring the derived mass into accordance with this relationship.

The visual magnitude of α Ursæ Majoris is 1·95 and the spectral type is Ko. Assuming a temperature of 4000°, we find from Table 15 of Eddington's *Internal Constitution of the Stars* that a correction of $-0^{m}{\cdot}60$ must be applied to the visual magnitude to reduce to the bolometric magnitude. For a series of values of the parallax, we can deduce the corresponding absolute bolometric magnitudes and then, from Table 14 of Eddington's book, the corresponding mass according to the mass-luminosity relationship can be inferred, account being taken of the temperature term as indicated at the foot of the table. For the same series of values of the parallax, the corresponding masses can be deduced, using the values of the period, semi-axis major of the relative orbit, and the mass ratio determined above.

The figures inferred in this way are given in Table VII.

TABLE VII

Comparison with Mass-Luminosity Relationship

Parallax	Absolute Bolometric Magnitude	Magnitude Corrected for Temperature Term	Mass from M.L. Relationship	Mass from Orbital Elements
″				
0·0200	−2·15	−2·38	9·50	11·34
·0225	−1·89	−2·12	8·40	7·96
·0250	−1·66	−1·89	7·62	5·69
·0300	−1·27	−1·50	6·48	3·37
·0350	− ·93	−1·16	5·64	2·14

The last two columns are in agreement for a parallax of about 0″·022. This value of the parallax corresponds to a mass of 8·6, and is in almost

exact agreement with the mean of the two trigonometric determinations. It appears improbable that the parallax can be as large as 0″·030.

The principal uncertainty that affects the determination of the parallax from the orbital elements is the value of $\sin i$. The values of $a \sin i$ in kms., deduced from the radial velocity measures, and the value of a in arc, deduced from the meridian and double-star observations, are both determined with considerable precision. If we accept the values of a (in seconds of arc) and of $a \sin i$ (in kms.) as correct, we can deduce the value of $\sin i$ that will give the parallax of 0″·022, inferred from the mass-velocity relationship. We derive in this way $\sin i = +0{\cdot}312$, $i = 161°{\cdot}8$.

Adopting this value of the inclination and using the formulæ :

$$A + G = 2a \cos(\omega + \Omega) \cos^2 \frac{i}{2},$$

$$A - G = 2a \cos(\omega - \Omega) \sin^2 \frac{i}{2},$$

$$B - F = 2a \sin(\omega + \Omega) \cos^2 \frac{i}{2},$$

$$-B - F = 2a \sin(\omega - \Omega) \sin^2 \frac{i}{2},$$

we can derive A, B, F, G by adopting the values of ω, Ω and a already determined.

The resulting values and the corresponding quantities inferred from the double-star observations are as follows :—

	Computed	Double-star Observations
	″	″ ″
A	+0·518	+0·519 ± 0·010
B	−0·358	−0·364 ± ·010
F	−0·334	−0·320 ± ·027
G	−0·495	−0·480 ± ·026

The constants computed with the inclination that satisfies the mass-luminosity relationship are therefore in agreement with the constants derived from the double-star observations within the assigned limits of error of the latter. It may be concluded that the discordance between the calculated mass and the mass deduced from the mass-luminosity relationship can be attributed entirely to uncertainties in the orbital elements that have been derived.

As mentioned above, the mass of the bright star, with the assumed parallax of 0″·022, is 8·6. The corresponding mass for the faint star is 3·84. The absolute bolometric magnitude (with assumed effective temperature of 5200°) corresponding to this mass on the mass-luminosity relationship is −0·10. The spectral type of the faint star is not known. Assume provisionally that it is of type M, with effective temperature 3000°. The corresponding bolometric magnitude is then +0·38. For an effective temperature of 3000°, this corresponds to an absolute visual magnitude of 2·09, or $3^m{\cdot}44$ fainter than the bright star. The corresponding apparent

visual magnitude is $5^m.39$. If the assumed spectral type is K2, the apparent visual magnitude becomes $4^m.90$. It is probably more reasonable to assume the type to be K2 than M, because an absolute visual magnitude of 2·09 appears low for a giant M-type star, and, in view of the mass that we have derived, it appears impossible for the faint star to be a dwarf. As mentioned in the introduction, there has been a wide range in the estimates of magnitude of the faint star, from the $4^m.89$ given by Kuiper to the $11^m.0$ given by Burnham. The magnitudes that have been given by double-star observers are only estimates, and the estimation of the brightness of a faint star in close proximity to a much brighter star are peculiarly liable to systematic error, the faint star being estimated to be fainter than it actually is. The only photometric measure is the one made by Kuiper, and is in almost exact agreement with the magnitude to be inferred from the mass-luminosity relationship, with assumed spectral type K2 for the fainter star. The present discussion is in support of the conclusion that the faint star is a giant, not a dwarf, of late type and that its apparent magnitude is much brighter than estimated by double-star observers.

VII. *Adopted Elements and Comparison with Observation*

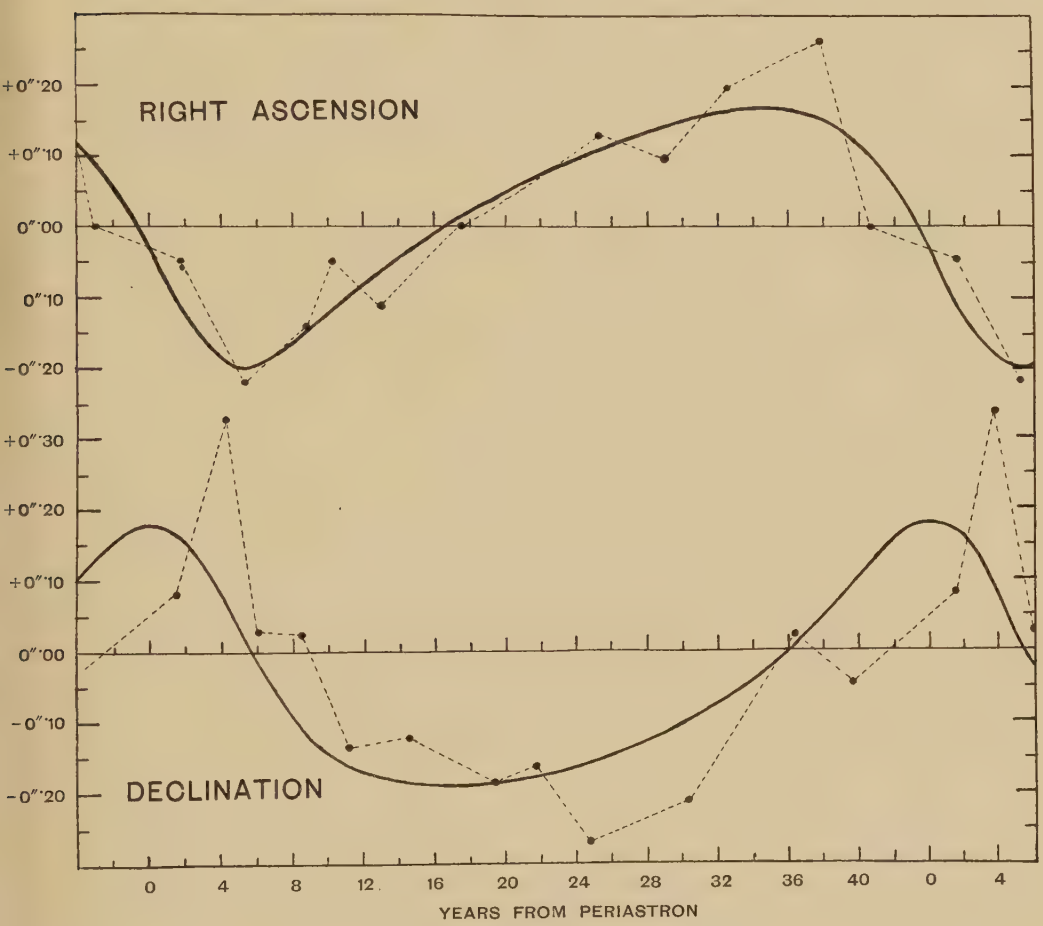
The following elements of the system are therefore adopted :—

		Orbit of Bright Star	Relative Orbit
Thiele-Innes Constants	A	+0″·160	+0″·518
	B	− ″·111	− ″·358
	F	− ″·103	− ″·334
	G	− ″·153	− ″·495
Semi-axis major		0″·196	0″·634
	ω	174°·0	
	Ω	139°·2	
	i	161°·8	
	P	44·0 years	
	T	1865·9	
Mass of bright star		8·6 × ⊙	
,, faint ,,		3·8 × ⊙	
Parallax of system		0″·022	
Velocity of centre of gravity		−8·72 km./sec.	
Semi-amplitude of velocity variation		2·01 ,,	

These final adjusted elements form the basis of the computed values of q and r for the meridian observations, given in Tables III and IV respectively ; of the computed separation and position angles for the double-star observations, given in Table V, and of the computed radial velocities, given in Table VI. The figure shows diagrammatically the representation of the meridian observations.

In the derivation of the final elements the double-star observations necessarily receive much greater weight than the meridian observations. The residuals given in the final columns of Tables III and IV are appreciably larger than those derived from elements based solely on the meridian observations. Whilst the residuals on right ascension are not unsatisfactory, those in declination show a systematic tendency. This is due to the fact

that the curve representing the change in declination displacement with time, derived from the meridian observations, differs in shape from that derived from the double-star observations. The discrepancy cannot be removed by change in eccentricity or in time of periastron. It is inherent in the observations. Double-star observations extending over a greater



Meridian Observations of α Ursæ Majoris

range in position angle are required in order to check whether the discordance arises from a systematic error in the double-star observations.

The residuals in position angle and separation given in Table V are not unduly large, except for the 1902–03 observations. As Aitken noted that his observation in 1902 was uncertain because of the difficulty of measurement, it is not unexpected to find that these residuals are large.

The principal difficulty in accepting the final elements occurs in connection with the failure to see the companion between 1905 and 1933. The computed position angles and separations, at intervals of five years, are as follows :—

Date	Position Angle	Separation
	°	″
1905·0	243·1	0·55
10·0	142·9	0·41
15·0	46·4	0·55
20·0	6·5	0·72
25·0	345·9	0·81
30·0	330·6	0·85

It would seem, therefore, that the companion should have been seen from about 1920 without any very great difficulty under favourable conditions. In reply to an inquiry about the negative observations at Lick, Dr. Aitken writes: "Possibly there were some nights of perfect seeing in that interval on which the companion might have been seen, but unfortunately I did not have the telescope directed on the star on such a perfect night. On the nights when I did examine it, I think I ought to have noted the companion at a distance of 0″·5 or even 0″·45, but with a smaller separation visibility would be very doubtful."

In view of this statement by Dr. Aitken, it would appear that our adopted elements give too large a separation from 1915·0 onwards. But there are difficulties in changing the elements so as to give smaller separations between 1915 and 1930. The apparent orbit can be closed up, without change of eccentricity, by adopting an earlier time of periastron passage. The result of such a change is to produce a considerable increase in the value of sin i, which leads to an impossibly large value of the parallax. The apparent orbit can also be closed up by increase in the eccentricity, but the meridian observations do not support such an increase; the best agreement with these observations is in fact obtained for an eccentricity of about 0·23. The eccentricity and time of periastron that we have adopted were derived by a process of trial and error, endeavouring to keep the eccentricity small and also to obtain a small value of sin i. For the adopted eccentricity, the time of periastron required to give a sufficiently small value of sin i is fixed within narrow limits.

Whatever elements are adopted, it remains impossible to account for the companion not being observed for some years previous to 1933. It was an easy object in that year, and it has remained an easy object since. The separation was changing very slowly in 1933, so that the separation must have been sufficient during several preceding years for observations to have been possible. The failure to see the companion at this time suggests that it may be variable in brightness. This may seem to be merely an *ad hoc* hypothesis. But in the case of the companion of Procyon, a similar hypothesis seems to provide the only possible explanation of the failure to see the companion from 1914 onwards. The orbit of Procyon is well determined, and in 1914 the separation was only slightly less than at the time of the discovery of the companion in 1896, when it was a relatively easy object.

The conclusion of this investigation is that it is not possible at the present time to derive elements that will fit all the available observational material (including the negative observations) satisfactorily, and which are at the same time in accordance with the mass-luminosity relationship. The

elements finally adopted are those which seem to us to be most free from objection. Of these elements, the period and the semi-axes majores of the bright star orbit and of the relative orbit are well determined; the adopted parallax, inclination and masses are not likely to be seriously in error. The time of periastron, the eccentricity and the angles ω and Ω are the elements which are most uncertain. An increase in eccentricity would make the meridian observations more discordant. For any adopted eccentricity, the time of periastron must be chosen so that the inclination agrees with the adopted value. The value of ω deduced from the meridian and double-star observations must be in satisfactory agreement with that deduced from the radial velocity observations. The uncertainties in the eccentricity and time of periastron would have been considerably reduced if double-star observations extending over a larger arc had been available. It is particularly unfortunate that there were no observations in the years preceding 1933. It is important that double-star observers should observe the star regularly each year, now that the separation has commenced to decrease, until observations are no longer possible. Photographic observations, to determine the motion of the bright star, will be continued at Greenwich.

VIII. *Ephemeris*

In Table VIII is given an ephemeris for the next ten years based on the final elements. In this ephemeris are given:

(*a*) The quantities $\Delta\alpha$, $\Delta\delta$ that are to be added to the right ascension and declination of the centre of gravity of the system in order to obtain the position of the bright star.

(*b*) The position angle, θ and distance ρ of the faint star relative to the bright star.

(*c*) The radial velocity of the bright star, V.

TABLE VIII

Ephemeris

	$\Delta\alpha$	$\Delta\delta$	θ	ρ	V
	s	″	°	″	km./sec.
1937·0	−·028	−·02	310·8	0·83	−10·65
38·0	−·025	−·05	307·7	·83	−10·60
39·0	−·022	−·08	304·4	·82	−10·53
40·0	−·017	−·10	300·9	·80	−10·45
41·0	−·011	−·12	297·2	·79	−10·35
42·0	−·004	−·13	293·1	·77	−10·23
43·0	+·004	−·12	288·6	·75	−10·08
44·0	+·011	−·10	283·7	·73	− 9·91
45·0	+·016	−·06	277·9	·70	− 9·70
46·0	+·019	−·01	271·5	·67	− 9·45
47·0	+·020	+·03	263·6	·63	− 9·16

Summary.—The available meridian, double-star and radial velocity observations of α Ursæ Majoris have been collected and discussed. Orbital elements have been derived, which give a general agreement with the observations. Attention is called to some discordances and the need for further observations of this interesting system.

VARIATION OF FAINT FRAUNHOFER LINES ACROSS THE SOLAR DISC.

M. G. Adam.

(Communicated by the Director of the University Observatory, Oxford)

The variation of line profile across the disc of the Sun provides one of the most sensitive tests of theories of absorption line formation. On account of the finite resolving power of the spectroscopes available for this work, such studies of profile cease to be significant for lines fainter than Rowland Intensity (R.I.) 4 or 5, and for such lines only the total absorption or equivalent width is an observable quantity.

It is the object of the present investigation to study the variation of equivalent widths of lines of R.I. -3 to $+5$ across the disc of the Sun. A well-marked regularity in centre to limb change in equivalent width is found. Further, there is a hitherto unsuspected * increase in equivalent width at the limb for the faintest observable lines. The first section of the paper deals with the observational determination of equivalent widths for these very faint lines, and the second section is concerned with an interpretation of the regularities thus found in terms of theories of line formation.

Section 1. *Observational Determination of Equivalent Widths*

The second order grating spectra (average dispersion 1·16 A. per mm.) used in the present investigation were obtained by H. H. Plaskett † with the large solar spectrograph of the Dominion Observatory, Ottawa. The original plates L. 4155, L. 4156, L. 4157 had been cut into three parts for microphotometry, and all measurements were made on the central section, which extended from $\lambda\lambda$ 5070–5160 A. Each of the three plates carried three spectra, one taken with a radial slit on the N. limb of the Sun, and two at the centre of the disc through a wedge erect and reversed. The wedge spectra not only provide a calibration for absolute intensity determinations, but also at some height give a spectrum from the centre of the disc comparable in intensity with the limb spectrum.

Visual Measurements.—In view of some uncertainty whether lines of R.I. -2 and -3 have ever been subsequently recorded, ‡ it is not surprising to find that comparatively few -2 and no -3 lines are included in equivalent width determinations. Thus in his "General Table," § Allen gives equivalent

* M. Minnaert, *B.A.N.*, **225**, 1931 ; G. Righini, *Oss. e Mem. Arcetri*, **51**, 1933 ; A. D. Thackeray, *Ap. J.*, 84, 405, 1936 ; *M.N.*, **97**, 672, 1937.

† *M.N.*, **91**, 870, 1931.

‡ A. Pannekoek, *M.N.*, 95, 732, 1935.

§ *Mem. Commonwealth Solar Obs.*, 5, 1934.

widths for only 13 −2 and no −3 lines in this region of the spectrum, while no −2 or −3 lines are included in his Table 6, or with any certainty in Mulders' similar calibration of the Rowland Intensity scale.*

In order to test whether such lines were recorded on the present plates, preliminary visual measurements of wave-length and estimates of intensity on the Rowland scale were made for all observable lines in the region λλ 5070–5160 A. Plates L. 4155 and L. 4156 were used for these measurements. The positions of all lines in the two centre spectra on each plate were measured on a Hilger micrometer, and converted into wave-lengths in the usual way by a linear formula and a correction curve based on lines of R.I. 5–0. The wave-lengths obtained from the four spectra on the two plates were compared and taken as referring to actual lines if measured on at least three spectra, the criterion of identity being agreement of wave-length within 0·10 A. In this way, in the region λλ 5070–5160 A., the mean wave-lengths of 310 lines of R.I. < 0 were determined for which the dispersion of the individual plate measures from the mean wave-length averaged ±0·009 A. In the same region 341 lines are given in the Revision of Rowland's Table with R.I. < 0, and of these 213 agree in wave-length within 0·05 A. with those now measured. These 213 lines are made up as follows:—

R.I.	Number of Lines	Average $\lambda - \lambda$ Rowland	Average Algebraic $\lambda - \lambda$ Rowland
−1	29	± ·0112 A.	+·0014 A.
−2	80	± ·0125 A.	+·0005 A.
−3	104	± ·0176 A.	−·0046 A.
	Mean	± ·0148 A.	−·0018 A.

There can therefore be no doubt that the present plates have sufficiently good definition for the detection and measurement of lines as faint as these.

Determination of Equivalent Widths.—The Oxford microphotometer has already been fully described.† Two magnifications are available with this instrument. After considering tracings made with both magnifications parallel to the dispersion, the low magnification was finally used, since for the −2 and −3 lines the profiles could be determined better for the sharper "peak." A slit ·025 mm. wide and < 0·4 mm. high was used in all the microphotometry. The plate was driven past the illuminating beam at a rate of approximately 7 cm. in 80 mins. The clear film to zero deflection was adjusted for each run to be as large as possible, and averaged approximately 10 cm.; its value was determined accurately for each tracing from clear film and zero readings put on at the beginning and end of each run.

The height in the spectrum at which tracings were made in both centre and limb spectra was chosen so that the galvanometer deflection for the faint

* *Zs. f. Ap.*, **8**, 63, 1934; **10**, 298, 1935.
† H. H. Plaskett, *M.N.*, **95**, 160, 1934.

lines was as large as possible.* In the centre spectra the value of cos θ was taken to be 1·00 for all the different heights ; in the limb spectra these most favourable positions corresponded to :

L. 4155	cos θ 0·28 approx.	average cos θ 0·29.
L. 4156	cos θ 0·19 ,,	
L. 4157	cos θ 0·39 ,,	

Ink lines had already been ruled on the plates for two values of cos θ,† and these were used in estimating the values of cos θ given above. The plates had to be covered on the microphotometer in two steps which overlapped in the wave-length region $\lambda\lambda$ 5132–5154 A. ; consequently in this region 6 instead of 3 measures were ideally available for determinations of wave-length and intensity. Characteristic curves at $\lambda\lambda$ 5082, 5114 and 5136 A. approx. were derived for each plate from tracings taken perpendicular to the dispersion in the two wedge spectra.

From these tracings and characteristic curves equivalent widths were determined by different methods for each of three main classes of lines. For the strong lines, half-profiles were plotted from each of the three plates for 26 lines at both centre and limb, and the areas measured with a planimeter. Thus for the line λ 5074·755 A. *Fe* (5), the profile was determined by about 20 measurements of line width, while for λ 5145·470 *Ti* (0) about 7 determinations of width were made. For the 23 of these lines of R.I. > -1 the deviation of any one plate from the mean equivalent width averaged 5·7 per cent. for the centre spectra measurements. For the second class of lines of R.I. < 0 it was impossible to measure more than the central intensity and a "base width," *i.e.* the base of a triangle having the same height and area as the line profile. For -2 and -3 lines this triangle generally coincided with the profile recorded on the tracing. Equivalent widths for 15 lines of R.I. < 0, as far as possible undistorted by blends or silver grain on the plates, were found in this way. The results of these direct equivalent width measurements for the 41 lines are shown in Table I. In the first three columns are given wave-length, identification and R.I. from the Revision of Rowland's Table. In the next three columns are given the equivalent width (and central intensity below and to the right) for each line on each of the three plates at the centre of the disc ; followed in column 7 by the mean equivalent width. Columns 8, 9, 10, 11 contain similar measurements for the lines at the limb.

Of the lines contained in Table I the equivalent width of the first 19 have been measured from profiles by Allen,‡ and 4 by Mulders.§ A comparison of their measures (abcissæ) and the present measures (ordinates)

* This meant that both centre and limb tracings were made from regions of the plate with approximately the same density. For the centre spectra from $\lambda\lambda$ 5070–5160 the mean height of the continuous background in units of the clear film to zero deflection was 0·35–0·07, and for the limb spectra 0·37–0·07.

† H. H. Plaskett, *M.N.*, **91**, 892, 1931.

‡ *Mem. Commonwealth Solar Obs.*, 5, 1934.

§ *Zs. f. Ap.*, **10**, 297, 1935.

TABLE I

Equivalent Widths at Centre and Limb for Stronger Lines

Rowland λ	Ident.	R.I.	Centre. E.W. in A. Central Int. cos θ 1·00 L. 4155	L. 4156	L. 4157	C. cos θ 1·00 Mean E.W.	Limb. E.W. in A. Central Int. cos θ ·28 L. 4155	·19 L. 4156	·39 L. 4157	L. cos θ ·3 Mean E.W.	L/C per cent.
Lines Measured by Profiles. Unblended Lines.											
5074·755	*Fe*	5	·110 ·467	·113 ·454	·121 ·455	·115	·098 ·580	·100 ·510	·110 ·545	·103	89·5
5081·121	*Ni*	3	·087 ·540	·091 ·522	·099 ·485	·092	·082 ·618	·077 ·623	·081 ·585	·080	87·0
5082·351	*Ni*	2	·054 ·626	·059 ·595	·069 ·579	·061	·065 ·675	·056 ·698	·061 ·674	·061	100·0
5083·347	*Fe*	4	·100 ·461	·109 ·405	·110 ·439	·106	·103 ·504	·103 ·462	·101 ·507	·102	96·1
5087·428	*Y*+	1	·046 ·696	·044 ·672	·051 ·633	·047	·047 ·718	·040 ·734	·047 ·697	·045	95·9
5094·420	*Ni*	0	·028 ·807	·035 ·767	·031 ·763	·031	·025 ·814	·029 ·805	·036 ·757	·030	96·8
5102·975	*Ni*	1	·042 ·726	·044 ·708	·045 ·711	·044	·049 ·784	·045 ·737	·049 ·709	·048	109·0
5109·657	*Fe*	2	·066 ·605	·068 ·585	·062 ·605	·066	...	·062 ·687	·061 ·663	·062	94·0
5110·409	*Fe*	5	·134 ·442	·136 ·398	·130 ·399	·133	·126 ·501	·128 ·482	·125 ·457	·127	95·5
5123·732	*Fe*	3	·100 ·496	·109 ·455	·103 ·458	·104	·105 ·528	·106 ·515	·099 ·514	·103	99·0
5127·370	*Fe*	3	·084 ·502	·099 ·458	·086 ·467	·090	·087 ·547	·098 ·528	·080 ·536	·088	97·8
5141·748	*Fe*	3	·071 ·552	·090 ·486	·083 ·512	·081	·074 ·590	·075 ·565	·089 ·583	·079	97·7
5145·470	*Ti*	0	·027 ·783	·034 ·759	·032 ·765	·031	·029 ·779	·037 ·760	·030 ·766	·032	103·2
5147·484	*Ti*	0	·027 ·769	·032 ·731	·027 ·768	·029	·036 ·721	·047 ·706	·032 ·737	·039	134·5
5151·919	*Fe*	3	·091 ·514	·096 ·447	·088 ·483	·092	·098 ·549	·092 ·522	·093 ·525	·094	102·2
5154·077	*Ti*+	2	·063 ·594	·077 ·528	·075 ·575	·072	·060 ·645	·069 ·618	·063 ·630	·064	88·9
5155·134	*Ni*	1	·037 ·662	·047 ·637	·047 ·689	·044	·035 ·755	·046 ·712	·038 ·731	·040	91·0
5155·773	*Ni*	2	·069 ·573	·081 ·490	·077 ·547	·076	·055 ·669	·071 ·624	·062 ·637	·062	81·5
5159·065	*Fe*	2	·063 ·594	·072 ·523	·068 ·584	·068	·054 ·662	·065 ·626	·060 ·644	·060	88·3
5113·449	*Ti*	0	·0218 ·812	·0193 ·828	...	·0205	·0247 ·800	·0246 ·831	·0251 ·827	·0248	120·8
5122·800	*Co*	−2	·0197 ·868	·0144 ·872	·0147 ·878	·0163	·0246 ·854	·0202 ·882	·0219 ·858	·0222	136·2
5136·103	*Fe*	−1	·0198 ·851	·0187 ·845	·0184 ·834	·0190	·0222 ·855	·0263 ·820	·0211 ·836	·0225	118·3

TABLE I—*continued*

Rowland λ	Ident.	R.I.	CENTRE. E.W. in A. Central Int. cos θ 1·00 L. 4155	L. 4156	L. 4157	C. cos θ 1·00 Mean E.W.	LIMB. E.W. in A. Central Int. cos θ ·28 L. 4155	·19 L. 4156	·39 L. 4157	L. cos θ ·3 Mean E.W.	*L/C* per cent.
Lines Measured by Profiles. Unblended Lines—contd.											
5157·985	*ZrNi*	−1	·011	·018	·015	·0146	·015	·021	·017	·018	123·1
			·864	·800	·871		·865	·850	·858		
5104·441	...	0	·028	·027	·029	·028	·028	·028	·027	·028	100·0
			·767	·790	·778		·784	·818	·811		
Blended Lines.											
5079·746	*Fe*	4	·093	·115	·123	·110	·103	·110	·106	·106	96·2
			·480	·423	·436		·551	·461	·520		
5139·475	*Fe*	4	·196	·214	·195	·201	·186	·197	·189	·191	95·0
			·388	·311	·361		·442	·429	·421		
Lines Measured by Triangles.											
5108·396	*Co*?	−1	·0139	·0088	·0096	·0118	·0165	·0176	·0088	·0143	121·1
			·871	·901	·874		·855	·874	·905		
5114·516	*La*+	−2	·0099	·0073	·0075	·0082	·0106	·0129	·0100	·0112	136·4
			·896	·923	·914		·893	·883	·895		
5146·778	*Co*	−2	·0073	·0107	·0091	·0090	·0089	·0115	·0103	·0102	113·2
			·924	·891	·901		·910	·892	·893		
5091·888	*Cr*	−1	·0101	·0076	·0134	·0104	·0136	·0156	·0146	·0146	140·1
			·908	·913	·863		·891	·859	·869		
5112·291	*Zr*+	−2	·0050	·0051	...	·0050	...	·0121	·0083	·0102	204·0
			·951	·939				·916	·916		
5148·684	*Ni*	−2	·0062	...	·0065	·0063	·0137	...	·0092	·0122	193·8
			·897		·920		·895		·904		
5149·797	*Co*	−2	·0074	...	·0064	·0069	·0113	·0086	·0084	·0095	137·9
			·909		·928		·902	·909	·910		
5091·726	...	−2	·0114	·0097	·0153	·0121	·0164	·0105	·0154	·0141	116·3
			·909	·894	·874		·888	·908	·880		
5100·658	...	−1	·0150	·0148	·0164	·0154	·0196	·0122	·0154	·0157	102·0
			·884	·870	·849		·867	·915	·865		
5124·617	...	−1	·0197	·0169	·0133	·0166	·0186	·0168	·0166	·0173	104·2
			·832	·851	·875		·859	·869	·854		
5132·672	...	−1	·0268	·0287	·0211	·0266	·0270	·0220	·0243	·0244	91·6
			·824	·784	·822		·844	·853	·835		
5140·825	...	−1	·0104	·0173	·0115	·0131	·0153	·0163	·0136	·0151	115·0
			·886	·867	·870		·861	·863	·882		
5143·734	...	−1	·0185	·0261	·0266	·0237	·0247	·0344	·0282	·0291	123·0
			·855	·809	·837		·818	·789	·838		
5148·842	...	−2	·0082	·0093	·0101	·0092	·0109	·0109	·0111	·0110	119·5
			·900	·886	·902		·897	·896	·898		
5150·196	...	−1	·0156	·0165	·0163	·0161	·0221	·0190	·0209	·0207	128·2
			·872	·865	·865		·847	·868	·863		
Measurements from Mg b Lines.											
5166·286	*Fe*	3				·101				·088	87·1
5171·612	*Fe*	6				·115				·0955	83·1
5172·700	*Mg*	20				1·128				·862	76·4
5183·621	*Mg*	30				1·61				1·17	72·6

is shown in fig. 1. It will be noted that the measurements compared with Allen's define with small dispersion a line with slope 0·987, showing a very good agreement between the two sets of measurements.*

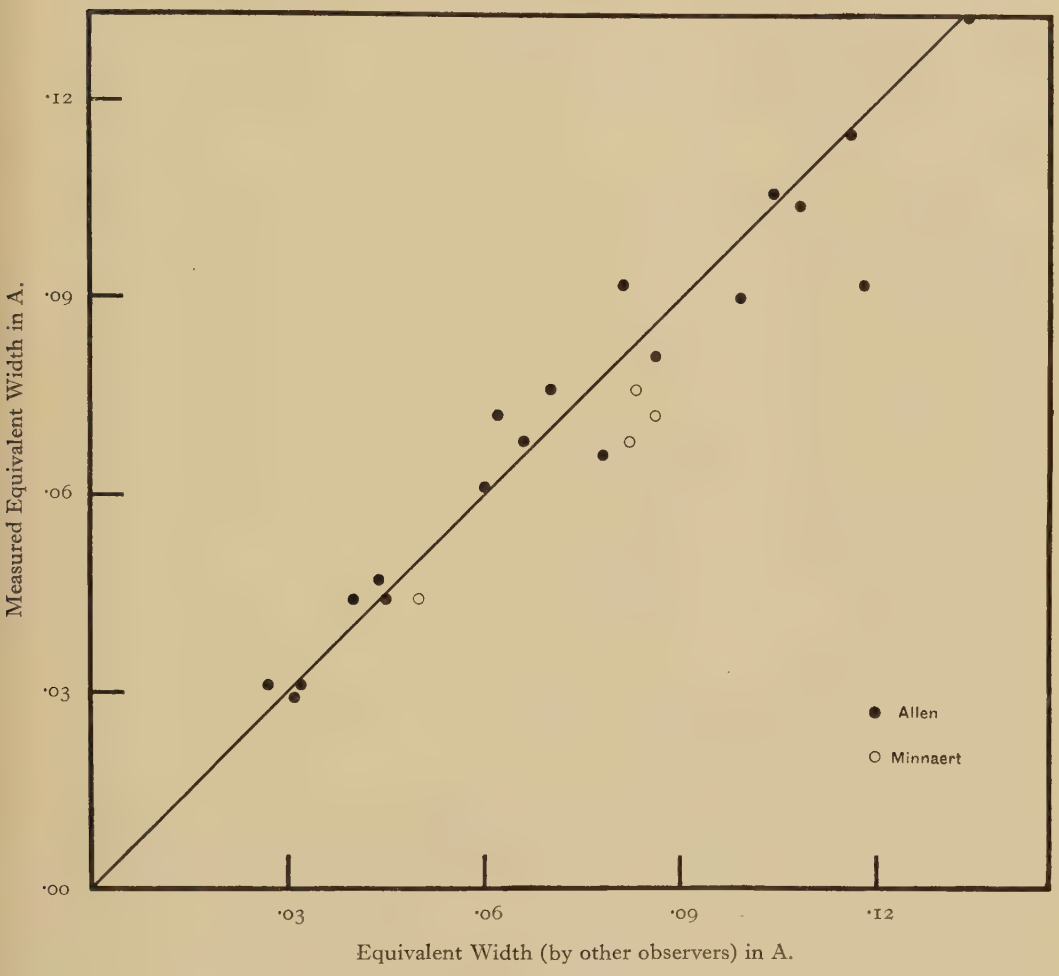


FIG. 1.—*Comparison of Equivalent Width measurements with those of other observers.*

Equivalent Widths for Faintest Lines.—For the faintest lines R.I. −2 and −3 we have not only the difficulty of measuring equivalent widths, but also the difficulty of being sure that we are measuring lines and not fluctuations due to silver grain. In order to overcome this latter difficulty, the central intensity (r) and wave-length of each small peak in turn was accurately measured over the whole wave-length region. The result was to obtain for both centre and limb spectra extended tables, of which a sample (for the centre spectra) is shown on the next page.

* R. Woolley, *Observatory*, 58, 329, 1935.

Centre Spectra, cos θ 1·00

L. 4155 λ	r	L. 4156 λ	r	L. 4157 λ	r	Mean λ	Mean r
5122·130	·900	5122·143	·905	5122·131	·901	5122·135	·902
5122·350	·940			5122·323	·937	5122·336	·939
				5122·441	·946		
				5122·573	·961		
5122·807	·870	5122·805	·872	5122·793	·875	5122·802	·872
				5123·029	·915		
5123·219	·803	5123·247	·802	5123·221	·798	5123·229	·801
5123·484	·851			5123·471	·871	5123·477	·871
5124·043	·868	5124·071	·900	5124·059	·889	5124·058	·886
5124·602	·832	5124·630	·851	5124·633	·875	5124·622	·852
5125·633	·915			5125·619	·923	5125·626	·919
				5125·840	·916		
5125·987	·913						
				5126·855	·966		
		5126·941	·925				
5127·695	·855	5127·692	·854	5127·694	·869	5127·694	·859
5127·871	·908	5127·868	·923	5127·885	·912	5127·875	·914
5128·136	·867	5128·104	·860	5128·106	·865	5128·115	·864
				5128·327	·944		
5128·386	·911						
		5128·500	·912	5128·488	·911	5128·494	·911
5128·561	·915						
		5128·898	·919				
5130·373	·844	5130·385	·798	5130·372	·382	5130·377	·825
5130·579	·893	5130·606	·904	5130·593	·889	5130·593	·895

A peak was then regarded as due to an actual line, if, and only if, peaks were measured on at least two plates with a difference in wave-length between the greatest and least not > 0·05 A. In this way the presence of 166 lines was established at the centre of the disc and 155 lines at the limb, all of R.I. 0, −1, −2, −3, of which just over 100 were common to both centre and limb. Those for which equivalent widths were to be measured were carefully selected from these. The basis of the selection was a comparison of the observed lines with Rowland's Table, and a detailed investigation of band lines, an account of which will be given in a later paper. Atomic lines were only selected which were not blended with carbon lines, and only the unblended molecular lines of the (0, 0) band. As a result 68 lines (35 atomic and 33 molecular) were available. Of the atomic lines 15 are given as measured by triangles in Table I. In order to find equivalent widths for the remaining 53 atomic and molecular lines, it was assumed that their profiles were wholly instrumental. A relation was therefore set up from the lines in Table I between measured equivalent width and central intensity (fig. 2), and by means of this relation and the mean central intensities (mean *r*) from the extended tables the Equivalent Widths given in Table II were determined.

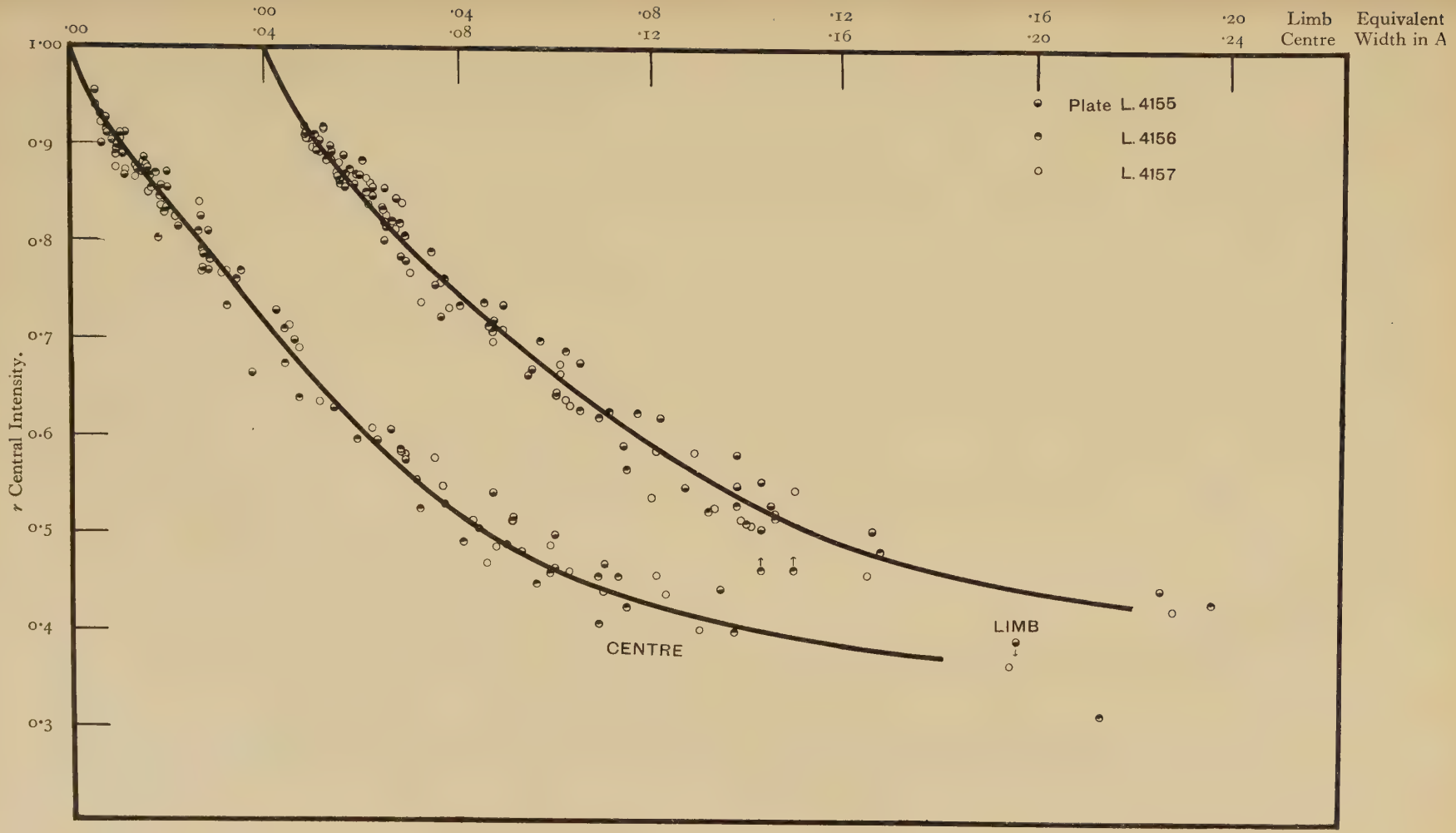


FIG. 2.—*Relation between Central Intensity and Equivalent Width.*

TABLE II

Equivalent Widths from Measured Central Intensities and Fig. 2

λ (Rowland)	Ident. (Rowland)	R.I.	Mean r Centre	Mean r Limb	E.W. in A. Centre	E.W. in A. Limb	L/C per cent.
Atomic Lines.							
5075·815	*Sc*	−3	·935	·931	·0052	·0067	128·8
5081·590	*Sc*	−2	·921	·913	·0070	·0091	130·0
5085·497	−*Sc*?	0	·837	·846	·0198	·0198	100·0
5086·623	...	−3	·895	·892	·0105	·0120	114·2
5094·947	*Co*	−1	·883	·884	·0122	·0132	108·0
5100·244	...	−3	·955	·941	·0030	·0054	180·0
5104·036	*Fe*	0	·800	·795	·0258	·0295	114·3
5104·198	*Fe*	0	·822	·804	·0222	·0274	123·3
5117·945	*Mn*	−1	·852	·832	·0172	·0221	128·5
5119·917	...	−3	·935	·933	·0052	·0063	121·1
5120·425	*Ti*	0	·750	·780	·0340	·0323	95·0
5122·131	*Cr*	−2	·902	·883	·0096	·0133	138·7
5123·222	Y^+	0	·801	·787	·0258	·0309	119·8
5123·473	*Cr*	−2	·861	·838	·0157	·0211	134·3
5127·690	*Fe*	−1	·859	·827	·0160	·0231	144·4
5135·188	...	−3	·939	·939	·0047	·0056	119·1
5138·723	...	−2	·885	·908	·0120	·0098	81·7
5145·740	...	−3	·943	·939	·0042	·0056	133·4
5156·072	*Sr*	−2	·902	·904	·0095	·0101	106·3
5157·209	...	−2	·898	·912	·0100	·0092	92·0
Carbon Lines.							
5073·464	−(*C* −)	−1	·902	·863	·0096	·0164	171·0
5081·768	(*C* −)	−2	·915	·912	·0078	·0092	118·0
5092·313	−(*C* −)	−1	·854	·856	·0168	·0179	106·6
5095·178	−(*C* −)	−1	·888	·867	·0115	·0159	138·1
5098·132	...	−1	·876	·852	·0135	·0185	137·0
5102·430	(*C* −)?	−2	·904	·877	·0092	·0143	155·5
5103·741	(*C* −)?	−2	·922	·902	·0071	·0105	148·0
5106·455	(*C* −)	−2	·891	·884	·0112	·0132	117·9
5109·124	(*C* −)?	−2	·938	·913	·0048	·0091	189·5
5114·264	(*C* −)	−2	·856	·880	·0166	·0139	83·7
5120·635	(*C* −)	−2	·919	·890	·0071	·0122	172·0
5132·676	...	−1	·810	·844	·0242	·0200	82·6
5135·585	(*C* −)	−2	·918	·913	·0072	·0091	126·2
5135·713	(*C* −)	−2	·920	·918	·0071	·0083	116·9
5136·276	(*C* −)	−2	·900	·885	·0098	·0130	132·7
5136·458	(*C* −)	−3	·925	·916	·0063	·0088	139·8
5138·112	(*C* −)	−3	·928	·946	·0059	·0049	83·2
5138·351	(*C* −)?	−3	·922	·910	·0071	·0095	133·9
5140·386	(*C* −)	−3	·938	·945	·0048	·0050	104·1
5141·219	(*C* −)	−2	·888	·881	·0114	·0138	121·0
5141·330	(*C* −)	−3	·910	·906	·0085	·0100	117·7
5142·112	(*C* −)	−3	·961	·907	·0022	·0099	450·0
5143·344	(*C* −)	−2	·922	·935	·0070	·0061	87·2

TABLE II—*continued*

λ (Rowland)	Ident. (Rowland)	R.I.	Mean *r* Centre	Mean *r* Limb	E.W. in A. Centre	E.W. in A. Limb	*L/C* per cent.
Carbon Lines—contd.							
5144·591	(*C* –)	–2	·873	·850	·0140	·0190	135·8
5147·106	(*C* –)	–2	·872	·896	·0141	·0116	82·2
5147·704	(*C* –)	–2	·899	·874	·0100	·0149	149·0
5151·461	...	–3	·943	·951	·0041	·0042	102·4
5154·412	...	–2	·832	·824	·0205	·0238	166·6
5156·656	–(*C* –)	–1	·821	·821	·0223	·0243	108·8
5157·616	(*C* –)	–2	·884	·859	·0122	·0172	141·0
5157·748	(*C* –)	–2	·889	·881	·0114	·0138	121·0
5158·534	(*C* –)	–2	·887	·872	·0117	·0151	129·0
(5158·609)			·898	·869	·0101	·0156	154·6

Change in Equivalent Width from Centre to Limb.—From Tables I and II we can derive the ratio $\frac{\text{equivalent width at centre}}{\text{equivalent width at limb}}$ for each line (hereafter denoted by *L/C*). In order also to obtain this ratio *L/C* for lines stronger than those included in the present wave-length region, the *Mg b* profiles * were measured with a planimeter, and these results are given at the foot of Table I.

The ratios *L/C* given in the last columns of Tables I and II are shown plotted against log *C* in fig. 3. The corresponding Rowland Intensities are also shown on the abscissa axis.† The surprising feature of the observations is that not only do the fainter Fraunhofer lines maintain their strength to the limb, but that there is an actual increase in equivalent width for lines of R.I. $< +0{\cdot}5$ approx.; an increase which rapidly becomes more pronounced with decreasing equivalent width at the centre. Although the observations necessarily show a very large scatter, increasing as would be expected with decreasing equivalent width, we may conclude that *L/C* increases from unity for lines of R.I. $+1$, -0, to 1·4 for lines of R.I. -3.

In view of the novelty of the present result it seems desirable to find some confirmation if possible in the work of other observers. In fig. 4 are plotted the ratios *L/C* against log equivalent width at the centre of the disc for some observations of Minnaert ‡ and of Thackeray. § It will be seen that their observations serve to fill in a gap in our fig. 3, and at the same time do indicate some increase of *L/C* with decreasing *C*. Righini || has also made some observations of equivalent width at centre and limb for lines of R.I. $\lessdot 0$, but the scatter of his observations is so large that they are not included in fig. 4, though they do show a somewhat similar trend.

* H. H. Plaskett, *M.N.*, **91**, 870, 1931.
† C. W. Allen, *Mem. Commonwealth Solar Obs.*, **5**, 22, 1934.
‡ *B.A.N.*, **225**, 1931. § *Ap. J.*, **84**, 405, 1936.
|| *Oss. e Mem. Arcetri*, **51**, 1933.

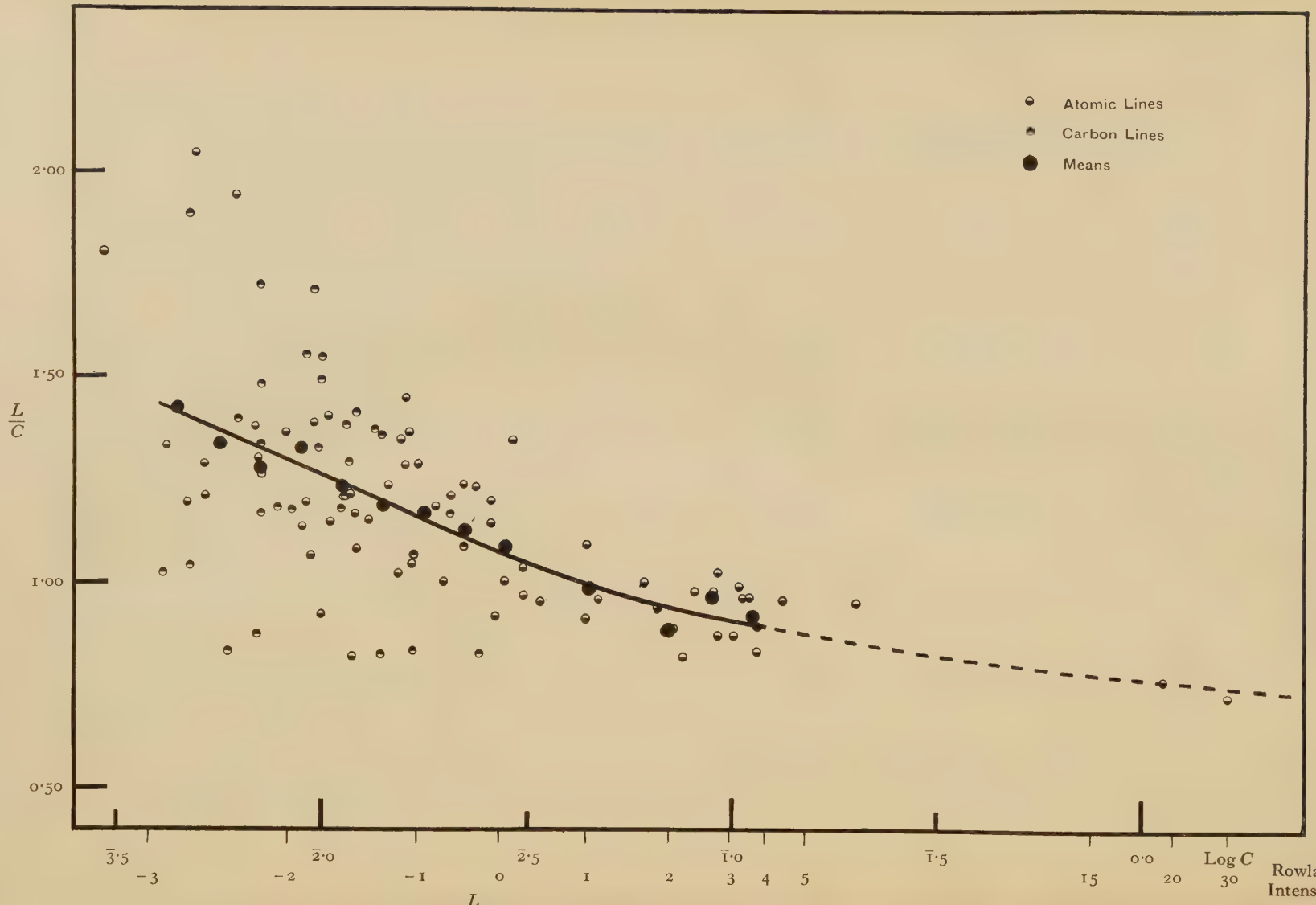


FIG. 3.—*Ratio* $\frac{L}{C}$ *against Log C.* (*The means are taken over intervals of* 0·1 *in Log C.*)

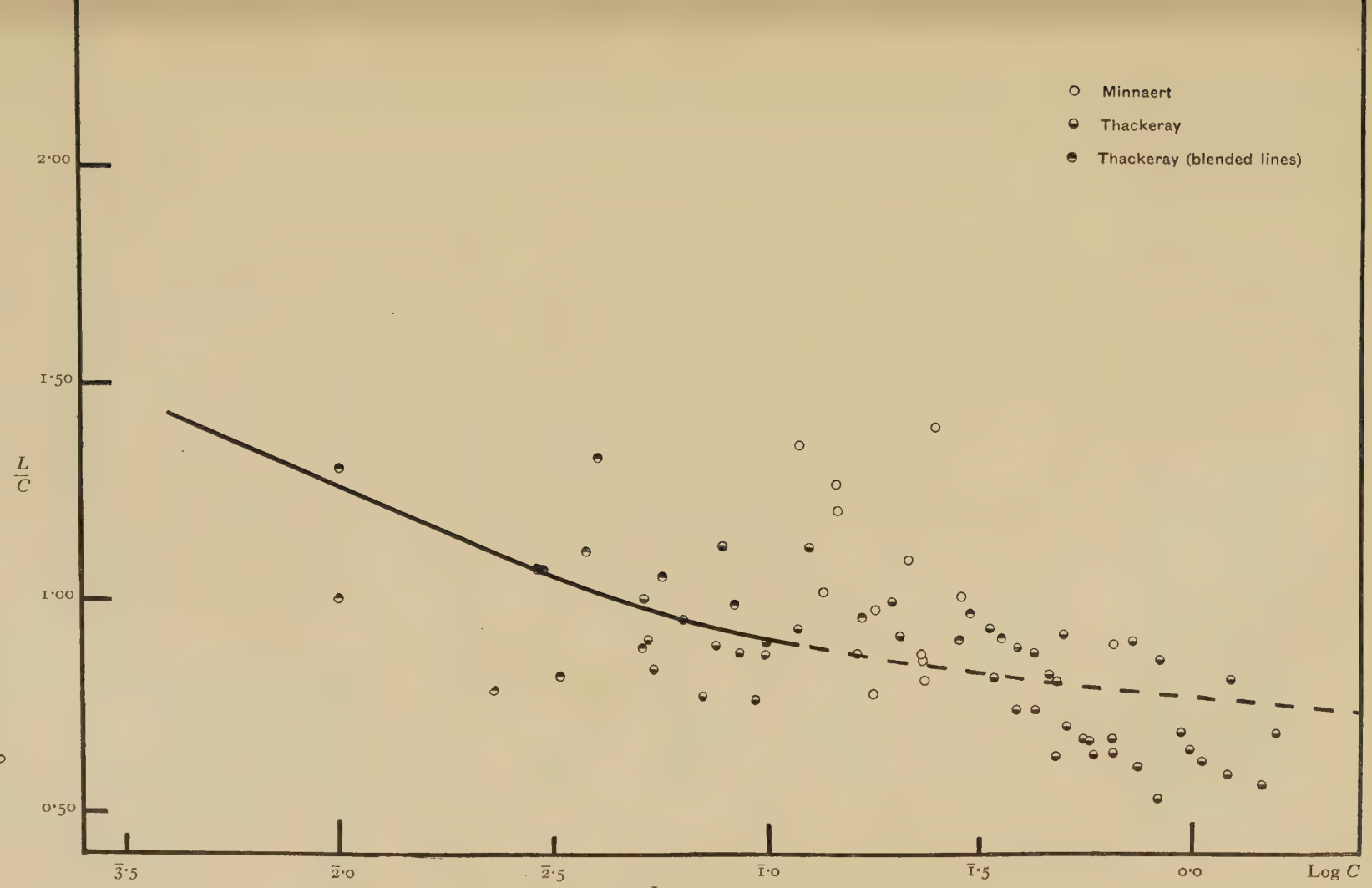


FIG. 4.—*Ratio* $\frac{L}{C}$ *against Log C, by other observers.*

There cannot therefore be much doubt that the effect shown in figs. 3 and 4 is real, and, it is important to note, less than that which must actually exist. If we suppose that the effect of scattered light in the Earth's atmosphere will be to superpose on the limb lines a scattered spectrum of dominantly disc centre light, this will tend to "fill in" the faint absorption lines at the limb. The faint limb lines in the absence of scattering in the Earth's atmosphere would therefore show an even greater ratio of L/C.

Section 2. Comparison of Observed Centre to Limb Changes in Equivalent Widths with Theoretical Predictions

It is well known that the "pure scattering" theory of the formation of absorption lines leads to their persistence across the disc, whereas the "pure absorption" theory predicts their complete disappearance at the limb, while a suitable combination of the two processes will give "any desired degree of contrast."* The problem now becomes more definite, in that we must not only choose our model atmosphere to fit the observed contrast at centre and limb for lines of one strength, but for the whole range of possible line strengths; in particular, and this is where the real difficulty occurs, we must provide for the increase of equivalent width of faint lines shown in fig. 3. In this section we consider the predictions of (1) pure scattering, (2) combined absorption and scattering, (3) a two-layer model atmosphere, and finally (4) the effect of reduced re-emission.

Atmospheric models lead to an expression for "r," the ratio of intensity at any wave-length in the line to that in the adjacent continuous spectrum, and "r" involves the atomic absorption coefficient a_ν. For the evaluation of a_ν we have used Menzel's expression 7·45,† calculated at λ 5100 A., $T = 6000°$, for the Ca^+ atom, both because it may be regarded as a typical element and because transition probabilities for the H and K lines have

Atomic Absorption Coefficient, Ca^+ at λ 5100, $T = 6000°$

$\Delta\lambda$ in A.	$a_\nu \times 10^{14}$	$\Delta\lambda$ in A.	$a_\nu \times 10^{18}$	$\Delta\lambda$ in A.	$a_\nu \times 10^{20}$
·000	347	·10	673	1·0	594
·004	340	·15	274	2·0	148
·010	301	·20	152	3·0	66
·020	200	·25	97·0	4·0	37·1
·030	101	·30	66·9	5·0	23·8
·040	38·4	·40	36·8	6·0	16·5
·050	11·98	·50	23·5	7·0	12·1
·060	2·68	·60	16·1	8·0	9·27
·070	0·560	·70	12·0	9·0	7·33
·080	0·167	·80	9·2	10·0	5·94

* H. H. Plaskett, *M.N.*, **91**, 913, 1931.
† *Lick Obs. Publ.*, **17**, 230, 1931.

been calculated for it by Zwaan. The integral was evaluated from the tables given by Mitchell and Zemansky.* The values obtained and used in subsequent calculations are given on p. 124.

1. *Pure Scattering.*—Since we are to account for strengthening in the case of weak lines, the pure scattering model atmosphere with a bounded photosphere is the most obvious one to try first. The equation of transfer in such an atmosphere is †

$$\cos\theta\frac{dI(\sigma,\theta)}{d\sigma}=I(\sigma,\theta)-\mathcal{J}(\sigma), \tag{1}$$

where $\sigma=\int_0^x \alpha_\nu N\rho dx$, and N is the number of atoms per gram. This leads to an expression for r.

$$r(\theta)=\frac{1}{2}\cdot\frac{1-(1/3)u_1}{1-u_1+u_1\cos\theta}\left[\frac{1+(3/2)\cos\theta}{1+\phi(\sigma_1)}+e^{-\sigma_1\sec\theta}\cdot\left\{\frac{2(1-u_1+u_1\cos\theta)}{1-(1/3)u_1}-\frac{1+(3/2)\sigma_1+(3/2)\cos\theta}{1+\phi(\sigma_1)}\right\}\right] \tag{2}$$

where $I(\sigma_1,\theta)$ has been expressed as $I(\sigma_1,0)(1-u_1+u_1\cos\theta)$, and $\phi(\sigma_1)$, an expression involving integral logarithms, replaces the $(3/4)\sigma_1$ in the Schuster equation. In this equation σ_1 is the total optical depth in the line frequency $=N_1\alpha_\nu$, where N_1 is the total number of atoms in the line of sight down to the photosphere. r values were calculated for this expression at different values of $\Delta\lambda$, with the appropriate σ_1 obtained from the table of α_ν by assuming a value for N_1. The line profiles thus obtained at centre and limb were measured with a planimeter, and so the predicted ratio L/C for any value of C determined with an accuracy of 1 or 2 per cent. The resulting L/C values for $u_1=\cdot67$, $\cos\theta=1\cdot0$ and $\cos\theta=\cdot3$ are shown in fig. 5. For small r values and $u_1>\cdot6$, this expression leads to an increase in r at the limb (lines become fainter). For large r values where the photospheric intensity is not reduced to zero at the line frequency, the increase in the absorption of the photospheric light over the longer limb path causes r (limb) to be $<r$ (centre), and there is an actual increase in the total absorption for weak lines and for the wings of strong lines.

The observations of Moll, Burger and Van der Bilt ‡ on the darkening to the limb in the continuous spectrum at this wave-length give the appropriate value of u_1 for any other value of $\cos\theta$, and the evaluation of r for other values of $\cos\theta$, shows that the strengthening of faint lines on the pure scattering model is extremely sensitive to position on the solar disc. The effect reaches a maximum at $\cos\theta=\cdot425$ approx., and for $\cos\theta<0\cdot25$ is replaced by weakening. The values of $\frac{L}{C}$ at $\cos\theta=\cdot4$ and $\cos\theta=\cdot2$ are also shown on fig. 5. Quite apart from this effect with $\cos\theta$, due in part to the linear approximation for $I(\sigma_1,\theta)$, the pure scattering model at $\cos\theta=\cdot3$

* *Resonance Radiation and Excited Atoms*, p. 321.
† H. H. Plaskett, *M.N.*, **91**, 908, 1931.
‡ *B.A.N.*, **3**, 83, 1925.

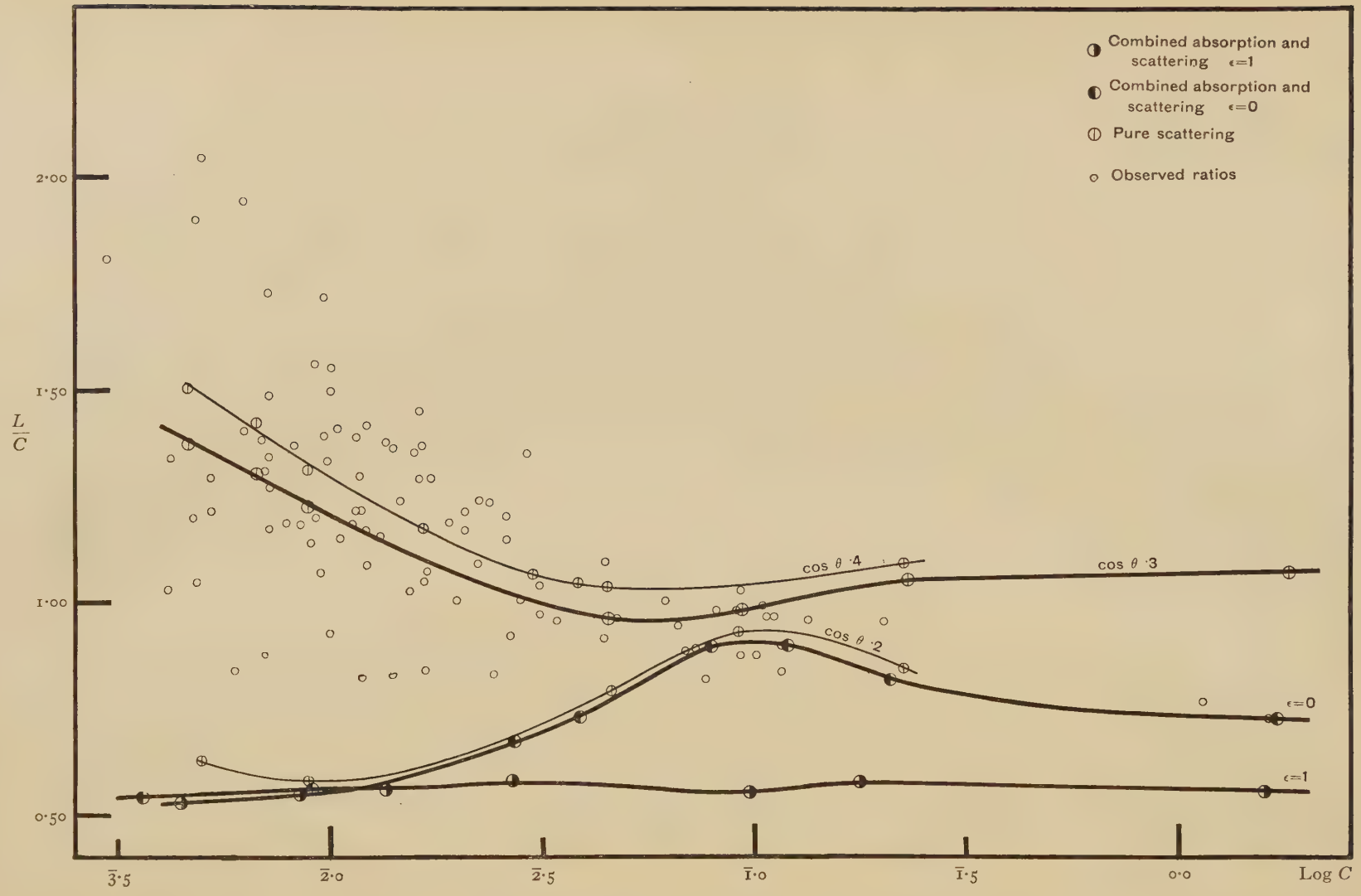


FIG. 5.—*Single Layer Atmosphere. Points show calculated ratios of* $\frac{L}{C}$ *against Log C.*

predicts strengthening for strong lines with well-marked wings, and must therefore be abandoned as not agreeing with the observations of fig. 3.

2. *Combined Absorption and Scattering.*—In order to fit the ratio $\frac{L}{C}$ for strong lines, the next step is to consider an atmosphere in which the continuous spectrum is formed all the way up, and in which we may introduce in the limits either a wholly scattering or a purely absorbing process in the line frequencies. In order to control more directly the ratio of scattering to absorption we have introduced Eddington's ϵ,* which in our notation $= \frac{k_\nu}{k_\nu + s_\nu}$, *i.e.* the ratio, constant over the line of the mass coefficient k_ν of line absorption, produced by collisions of the second kind, to the total atomic coefficient of absorption $k_\nu + s_\nu$. The equation of transfer is now

$$\cos\theta \frac{dI(x_1\theta)}{\rho dx} = (k_0 + k_\nu + s_\nu)I(x_1\theta) - (k_0 + k_\nu)B(x) - s_\nu \mathcal{J}(x). \qquad (3)$$

With the substitutions $B(\tau) = a + b\tau$, and

$$\xi^2 = \frac{k_0 + k_\nu}{k_0 + k_\nu + s_\nu} = \frac{1 + \epsilon K\alpha_\nu}{1 + K\alpha_\nu},$$

$$n = \frac{k_0 + k_\nu + s_\nu}{k_0} = 1 + K\alpha_\nu,$$

where k_0 is the coefficient of continuous absorption and where $K = \frac{N}{k_0}$ is a quantity needed for numerical calculation, we obtain, when ξ^2 and n are constant with depth, the following expression for r :—

$$r(\theta) = \frac{a + \frac{b}{n}\cos\theta + \frac{\sqrt{3}}{2}\,\frac{\frac{2}{3}\cdot\frac{b}{n} - a}{\xi + \sqrt{3}/2}\,\frac{1 - \xi^2}{1 + \xi\sqrt{3}\cos\theta}}{a + b\cos\theta}. \qquad (4)$$

If we take $\epsilon = 0{\cdot}5$ for all lines we find a ratio $\frac{L}{C}$ of 61 per cent. for strong lines, which rises to 66 per cent. for lines of medium strength, and falls again for fainter lines. An obvious refinement of this simple model is to take a smaller value of ϵ for the strong lines than for the weak lines ; since the average optical depth for a strong line is less than that for a weak line, it follows that the pressure is less and so ϵ is less. The effect of changing ϵ in (3) is chiefly upon the central part of the line, so that while L/C for the strong lines increases from 55 per cent. for $\epsilon = 1$ to 72 per cent. for $\epsilon = 0$, the effect on the faintest lines is almost inappreciable. L/C ratios for $\epsilon = 0$ and $\epsilon = 1$ are shown in fig. 5, and it will be seen that there is no question of a fit with observation for lines weaker than $C = {\cdot}1$ A. approx.

3. *Double Layer Atmosphere.*—In the actual solar atmosphere there must be an outer region in which collisions become negligible, so that we

* *M.N.*, 89, 622, 1929.

have a layer at the top in which no continuous spectrum arises. We may therefore make a closer representation of reality by a "double layer" atmospheric model, in which a combined absorbing and scattering layer A is surmounted by a pure scattering layer B. The principle of calculating the residual intensity in a line at any frequency in such an atmosphere, is to find the intensity emerging from layer A and its variation with θ, *i.e.* the appropriate coefficient of darkening u_ν at this frequency and allow this emergent intensity to traverse the pure scattering layer B. The effect of the superimposed layer B is to alter the boundary conditions at the top of layer A, from those assumed in the single layer atmosphere (no back

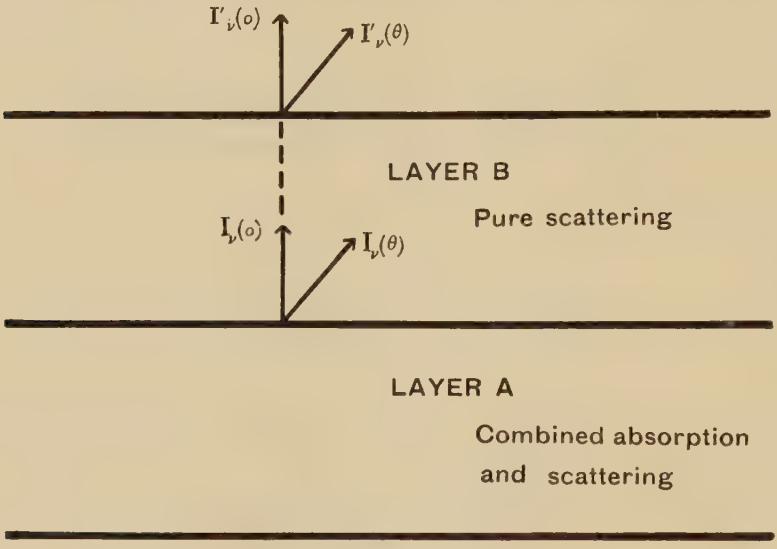


FIG. 6.—*Double Layer Atmosphere.*

radiation). In order to avoid arithmetical complications which would result in calculating line profiles if this new boundary condition were taken into account, we have assumed $\epsilon = 1$, so that layer A becomes purely absorbing. In other words, we assume that all incident radiation that excites atoms, including that coming from layer B, is converted by collisions of the second kind into thermal agitation, and simply warms up this layer. In such a model the effect of the backward radiation shows itself only in altering the constants a and b of the formula $B(\tau) = \frac{\sigma}{\pi} T^4 = a + b\tau$, and these are in any case determined from the observed darkening towards the limb in the continuous spectrum.

The intensity emerging from layer A at an angle θ is given by

$$I_\nu(\theta) = a + \frac{b}{n} \cos\theta \qquad (5)$$

(*cf.* equation (4), $\epsilon = 1 \therefore \xi^2 = 1$), and $\frac{I_\nu(\theta)}{I_\nu(o)} = 1 - u_\nu + u_\nu \cos\theta$ gives the

appropriate u_ν for layer B. The intensity $I_\nu'(\theta)$ then emerging in the line from layer B is given by (*cf.* equation (2))

$$I_\nu'(\theta) = I_\nu(o)(1-(1/3)u_\nu)\left[\frac{1+(3/2)\cos\theta}{1+\phi(\sigma_1)} + e^{-\sigma_1 \sec\theta}\right.$$
$$\left.\left\{\frac{2(1-u_\nu+u_\nu\cos\theta)}{1-(1/3)u_\nu} - \frac{1+(3/2)\sigma_1+(3/2)\cos\theta}{1+\phi(\sigma_1)}\right\}\right]. \quad (6)$$

This expression divided by $a+b\cos\theta$, the continuous spectrum intensity emerging from A and unaffected by the scattering layer B, gives $r_\nu'(\theta)$.

In calculating profiles on this model we must assume a value for $K\left(=\frac{N}{k_0}\right)$ in layer A and for N_1 in layer B. In all calculations we have kept the ratio $\frac{N_1}{K}$ constant for lines of all strengths, *i.e.* in effect we have assumed no stratification between elements responsible for the faint lines and those responsible for the strong lines. Once again we find that the strong and faint lines cannot be fitted simultaneously. If we take $N_1 = 10K$ the faint lines are moderately well fitted (fig. 7), but the ratio of L/C for the strong lines is 99 per cent. If, on the other hand, we take $N_1 = \frac{K}{10}$ we can reproduce the centre to limb change for the strong lines, but are quite unable to account for the behaviour of the faint lines.

We must bear in mind that whereas in this double layer atmosphere N_1 is simply a number of atoms in the line of sight, K is the ratio of a number of atoms per gram to k_0. A change in this ratio due to the higher layers in which the strong lines must chiefly originate may be justified. For any individual element the number of atoms in the lower state of excitation must decrease with depth on account of the higher temperatures, so that N is smaller at great depths; further, k_0 being due to all atoms will not show this falling off, but will rather increase with depth on account of the increasing pressure. We may therefore increase the ratio $K = \frac{N}{k_0}$ for strong lines without sacrificing the assumption of a completely mixed atmosphere, but to fit the observations we should need an increase of 100 in the ratio between lines of Rowland Intensity +1 and +3. The only alternative is that 99 per cent. is the limiting value of the ratio L/C for which the absorption lines can be treated statistically, and that for lines stronger than centre equivalent width 0·1 A. we must consider the individual peculiarities of the atoms concerned.

4. *Reduced Re-emission.*—Since neither the single layer nor the double layer model atmosphere predicts the observed centre to limb change in equivalent width for both strong and faint lines, we must seek some powerful mechanism for strengthening lines to the limb which will operate differently upon lines of different strengths. It is obvious that absorption at any frequency without subsequent re-emission in that frequency will produce a great increase in line strength when we consider the longer path taken by

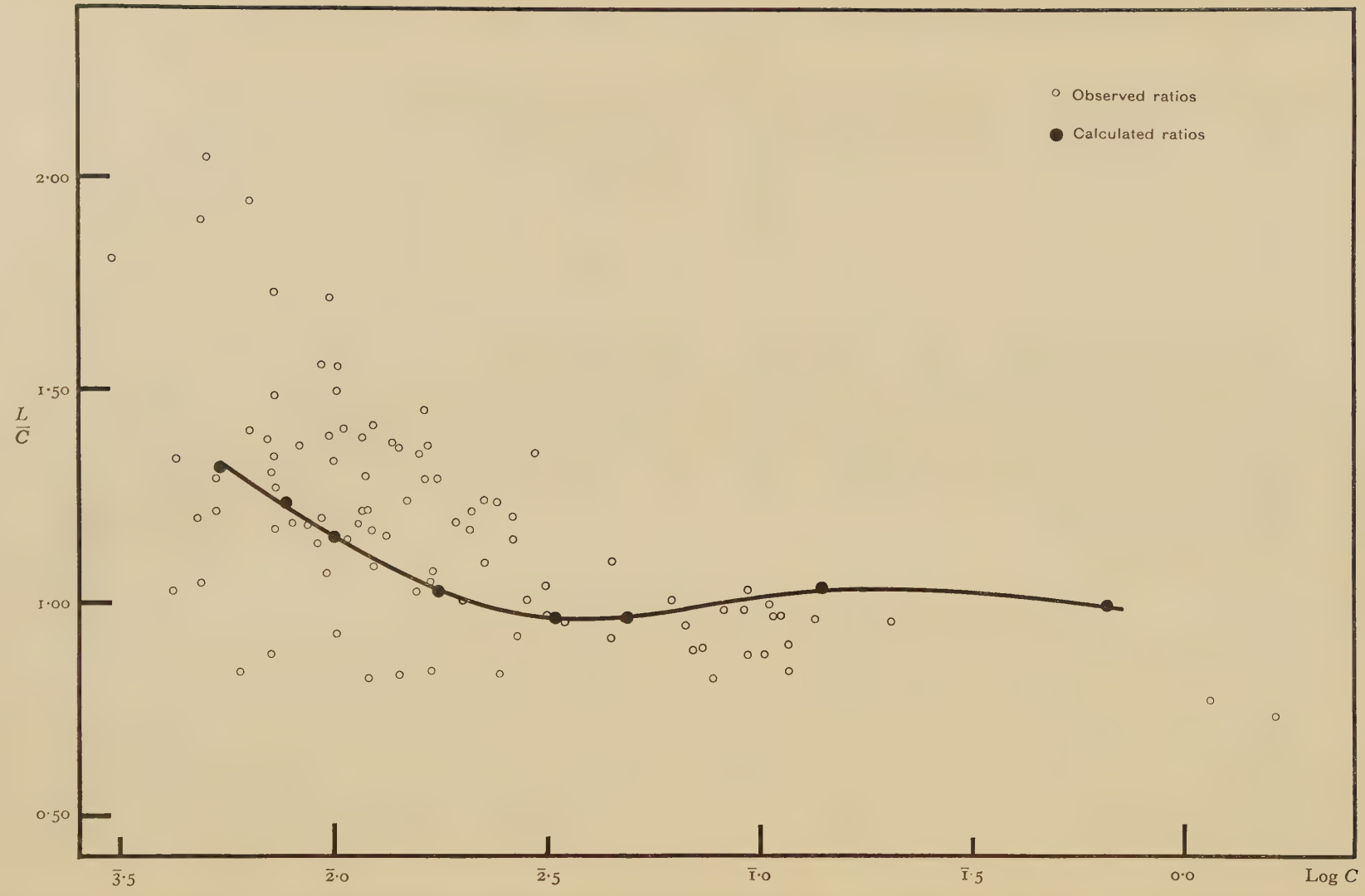


FIG. 7.—*Double Layer Atmosphere. Calculated ratios of* $\frac{L}{C}$ *against Log C with* $\frac{N_1}{K}(=10)$ *to fit the faint lines.*

the light which is observed at the limb. Let us now examine the consequences of assuming such an effect to operate on faint lines.

We may evaluate the effect of reduced re-emission numerically by considering that the emission is reduced by the factor β where β varies between 0 and 1. If we consider the extreme case of $\beta = 0$, the desired strengthening to the limb cannot be obtained in a single layer Eddington model with combined absorption and scattering since it is easy to show that equation (4) becomes

$$I_\nu(\theta) = \xi^2\left(a + \frac{b}{n}\cos\theta\right),$$

leaving the ratio $\frac{r_L}{r_C}$ unaltered. In the double layer atmosphere β has no effect in the first layer, since by assumption $\epsilon = 1$, *i.e.* there is no selective re-emission in the line frequency. In layer B, however, the equation (1) of transfer becomes

$$\cos\theta\,\frac{dI(\sigma,\theta)}{d\sigma} = I(\sigma_1\theta) - \beta\mathcal{J}(\sigma),$$

which on integration with $\beta = 1$ gives equation (6), and with $\beta = 0$ gives

$$I_\nu'(\theta) = \left(a + \frac{b}{n}\cos\theta\right)e^{-\sigma_1\sec\theta} \qquad (7)$$

or exponential absorption in layer B.

Ratios of L/C from profile calculations with $\beta = 0$ for lines fainter than centre equivalent width 0·1 A. approx., and $\beta = 1$ for lines stronger than this, are shown in fig. 8. For k_0 constant and keeping the ratio $N_1 = \frac{K}{10}$ which gives a fit for the strong lines, we get only 100 per cent. for L/C for faint lines. Now, however, an increase in $K = \frac{N}{k_0}$ of 2 or 3, as compared with the factor 100 needed previously in § 3, is sufficient to give a ratio of L/C for the faintest lines of 135 or 150 per cent. It will be seen that the upper curve in fig. 8, in which K for the faint lines varies from one to three times its value for the strong lines, gives a very good fit with the observations. Profiles for the faintest and strongest lines at the interface of layers A and B, and at the top of layer B, from which the fitted curve of fig. 8 is derived, are shown in fig. 9. These show the large increase in L/C produced by introducing reduced re-emission in the second layer. These profiles also show that 71 per cent. of the faintest lines is still formed in the first layer, and, moreover, as we have pointed out before, the average level of formation in this layer must be deeper than for the strong lines ; hence there is no essential contradiction with the observations of Evershed effect in the Sun or of solar rotation, both of which point to a deeper level of formation for faint lines.

It appears, then, that a reduction in re-emission relative to absorption for faint lines will give the observed strengthening to the limb. The

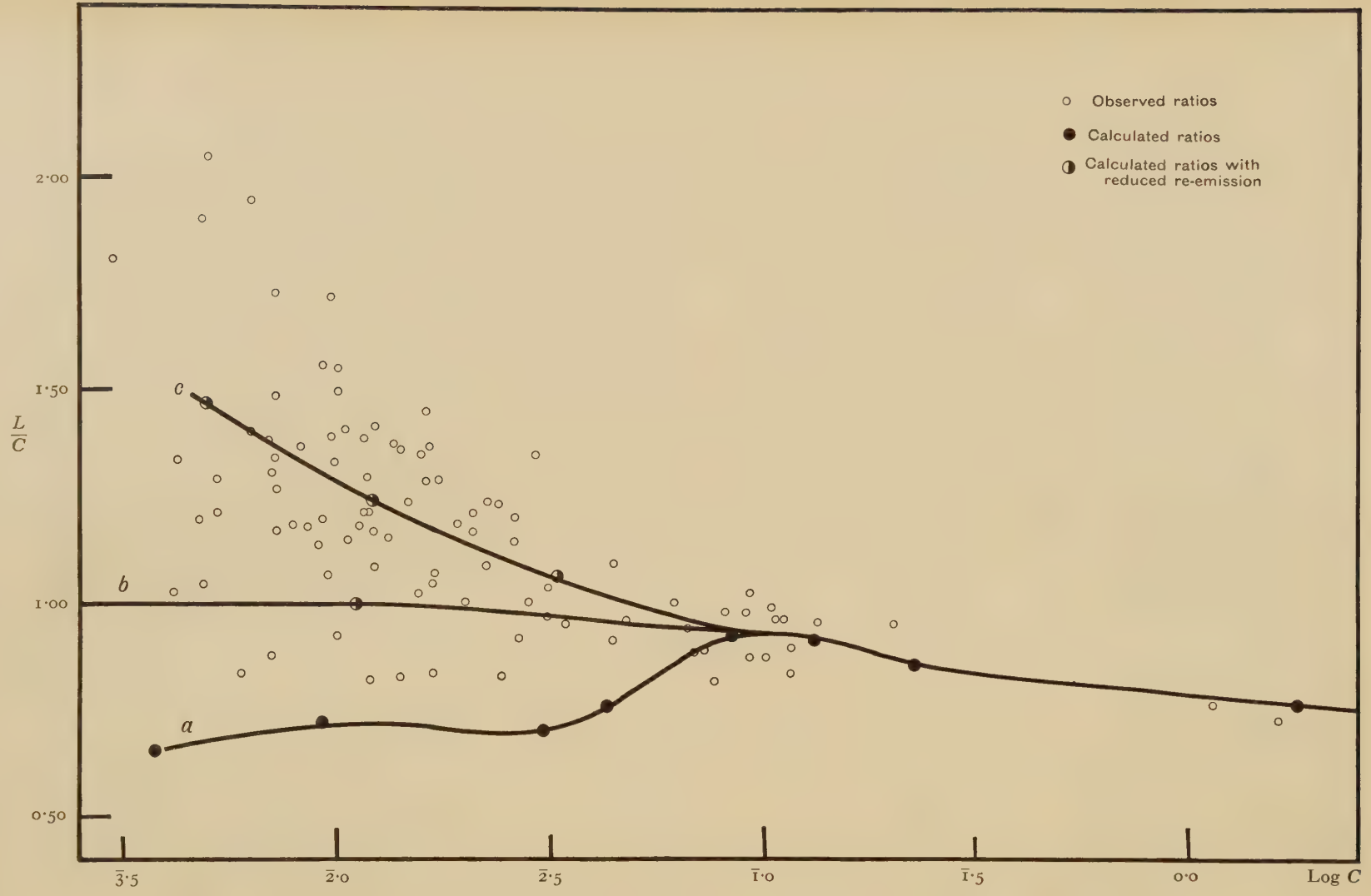


FIG. 8.—*Double Layer Atmosphere. Calculated ratios of L/C against Log C with* $\frac{N_1}{K}(=10^{-1})$ *to fit the strong lines. Curve (a) normal re-emission, curve (b) reduced re-emission, curve (c) reduced re-emission with* $\frac{N_1}{K}$ *increased for faint lines.*

question then arises whether there is any mechanism which operates for faint atomic lines, as opposed to strong atomic lines which will tend to reduce their re-emission. It seems reasonable to suppose that the interlocking described by Woolley * will act in this way. Consider the very simplest case of such interlocking provided by two levels α and β and a common upper level γ, as in fig. 10 (*a*). Let the statistical weights of α, β be g_α and g_β. Then from

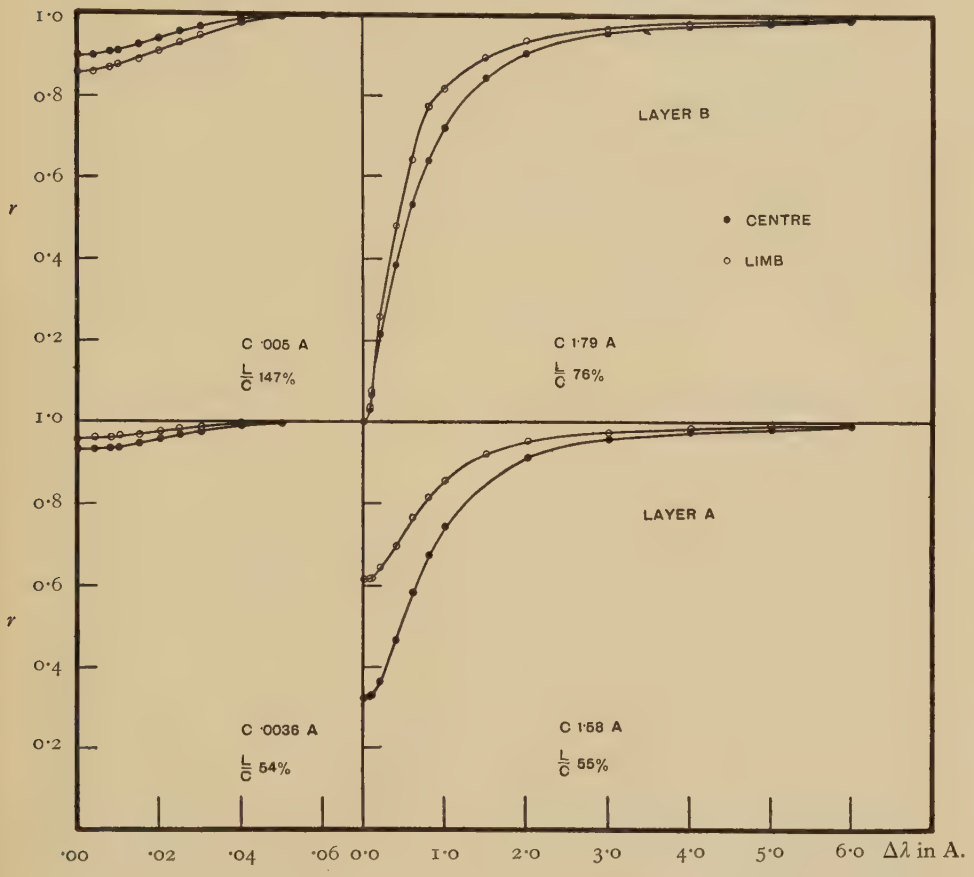


FIG. 9.—*Calculated profiles of strong and faint lines at centre and limb at the interface of layers A and B, and at the top of layer B. The lines shown are the strongest and faintest from the curve (c) of Fig. 8.*

the sum rule the transition probabilities will be given by $A_{\gamma\alpha} : A_{\gamma\beta} = g_\alpha : g_\beta$, and $B_{\alpha\gamma} = B_{\beta\gamma}$. We shall assume $g_\alpha >> g_\beta$. For uniform incident radiation a steady state is maintained, and the atomic populations of α and β are as their statistical weights. At any level in layer B, however, the incident radiation will not be uniform, but will be crossed by a strong absorption line in the frequency $\nu_{\alpha\gamma}$ and a weak absorption line at the frequency $\nu_{\beta\gamma}$. Let the residual intensity at $\nu_{\alpha\gamma}$ be ω of that at $\nu_{\beta\gamma}$, where ω is less than one. Then if the population in α and β are maintained in the ratio of

* *M.N.*, **90**, 779, 1930 ; *Ap. J.*, **73**, 194, 1931.

their statistical weights, the transitions up from α and β will be in the ratio $\frac{g_\alpha \omega B_{\alpha\gamma}}{g_\beta B_{\beta\gamma}} = \frac{g_\alpha}{g_\beta} \cdot \omega$, and the transitions down from γ in the ratio $\frac{g_\alpha}{g_\beta}$. This means that more atoms pass from β–γ than from γ–β, the excess downward transitions going from γ–α, and there is the required reduced re-emission in the faint line frequency $\nu_{\beta\gamma}$.

It is important to note, however, that interlocking will produce the required reduction in emission in the faint line frequency, if, and only if, the atomic populations in α and β are kept in the ratio of their statistical weights by some external mechanism. If the atom had only the three states α, β, γ, and there were no external mechanism such as collisions, the effect of the reduced re-emission would be simply to redistribute the atoms between α and β until a steady state was set up, and there were as many transitions up as down in both lines. The new populations in the levels α and β would then be n_α, n_β, where

$$\frac{n_\alpha \omega B_{\alpha\gamma}}{n_\beta B_{\beta\gamma}} = \frac{A_{\gamma\alpha}}{A_{\gamma\beta}} = \frac{g_\alpha}{g_\beta},$$

or

$$\frac{n_\alpha}{n_\beta} = \frac{1}{\omega} \cdot \frac{g_\alpha}{g_\beta}.$$

Any mechanism which, for example, makes $\frac{n_\alpha}{n_\beta} > \frac{1}{\omega} \cdot \frac{g_\alpha}{g_\beta}$, will reduce the re-emission in the strong line and increase it in the faint line.

If collisions were effective we should have a mechanism by which the populations in α and β were maintained in the ratio of their statistical weights, but in the assumed absence of collisions in layer B we can only suppose that on the average other atomic lines do maintain the atomic population in roughly the ratio of the statistical weights. In the case of actual atoms the situation is of course much more complicated. There will be, however, a statistical tendency for faint lines produced as a result of low transition probability upwards to show reduced re-emission (provided always that the atomic populations are maintained in the ratio of the statistical weights), and to this extent we may assume that interlocking is responsible for the reduced re-emission in faint atomic lines.

If this interpretation is correct it requires a difference in behaviour for lines faint on account of small transition probability upwards, and those faint from low relative abundance; thus faint Rare Earth absorption lines should not be strengthened at the limb, since for these there will be the ordinary re-emission in the faint line frequency.

Unfortunately, as Tables I and II show, the lines now measured are all due to the more abundant elements. The only remaining test is to consider the transitions responsible for the lines. Of these latter four are singlet lines,* for which there cannot be any interlocking within the multiplet of the type now considered. The difference between the ratio L/C

* C. E. Moore, *A Multiplet Table of Astrophysical Interest*, 1933.

measured for these lines and the mean L/C corresponding to their equivalent width is given below.

λ	Ident.	Transition	Measured L/C − Mean L/C
5081·121	*Ni*	$z'F^\circ - e'G$	− 4%
5155·773	*Ni*	$z'D^\circ - e'F$	−11%
5120·425	*Ti*	$a'H - z'I^\circ$	− 8%
5123·222	Y^+	$a^3F - z'P^\circ$	+12%

The three lines due to neutral elements do behave as we should expect, but it must be emphasized that for these faint lines where the observational error is necessarily so large a great number of such singlet lines would be

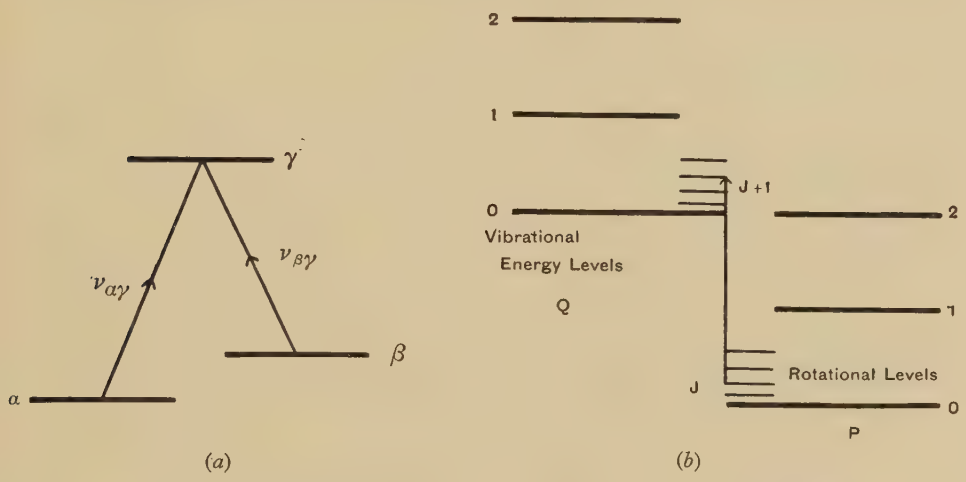


FIG. 10.—*Schematic Energy Level Diagrams for* (*a*) *atoms*, (*b*) *molecules.*

required to give a real test of the predictions. Further investigation of this point and the measurement of lines faint on account of low relative abundance must await future study proposed with the new solar spectroscope. In the meantime it may be pointed out that Menzel's * observation that the Rare Earth lines in the flash spectrum pass over into emission at a much earlier stage than ordinary absorption lines of the same strength, is some evidence for the present interpretation.

We must remember that in fig. 3 the increase in the ratio L/C with decreasing C is also observed for molecular lines. This requires that some mechanism must also exist for them by which re-emission is reduced relative to absorption. Consider a band arising as a result of the transition of the molecule between two electronic states P and Q, as in fig. 10 (*b*), so that the λ 5165 band is a result of a transition between the (0, 0) vibrational levels,

* *Publications Lick Obs.*, **17**, 299, 1931.

while the individual lines of the band correspond to a change in rotational quantum number. Once the molecule is in the state $Q_0(\mathcal{J}+1)$, its collisional area, already from kinetic theory larger than that of the atom, will be further increased by its being in an excited electronic state.* Thus collisions of the second kind for excited molecules in layer B will be more frequent than for excited atoms, and these collisions may result either in de-excitation back to $P_0(\mathcal{J})$ or in a further successful excitation to some of the closely spaced levels above $Q_0(\mathcal{J}+1)$. In either case re-emission in the rotational line will be reduced, and we shall tend to get the observed strengthening of molecular lines to the limb.

We may therefore conclude, pending further observational and theoretical studies, that reduced re-emission in the upper layers of the solar atmosphere, some theoretical justification for which has been offered, will account for the observed behaviour of faint lines towards the limb of the Sun.

Summary.

1. Equivalent widths at centre and limb are measured for 94 lines of Rowland intensity 5 to −3.

2. It is shown that the ratio of limb equivalent width to centre equivalent width increases regularly with decreasing centre equivalent width, and reaches a value of 1·4 for the −3 lines.

3. The predictions of existing theories of line formation are examined and found to be unable to account for this strengthening.

4. A "double layer" atmosphere is considered which also fails to fit the observations.

5. The assumption of reduced re-emission in the double layer atmosphere is shown to predict the required strengthening.

6. The interpretation of reduced re-emission for atoms and molecules is considered.

It is a great pleasure to acknowledge the constant help and encouragement of Professor H. H. Plaskett at every stage of the investigation, and to point out that it is only the excellence of his observational material which made the faint line measurement possible. I should also like to thank Dr. R. O. Redman for reading the paper and for some helpful suggestions.

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* Ruark and Urey, *Atoms, Molecules and Quanta*, i. 498.

ON THE MASSES IN SINGLE-SPECTRUM BINARY SYSTEMS.

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1. In his excellent book, *The Binary Stars*, Dr. Aitken says:* "The value of the function $m_1^3 \sin^3 i/(m+m_1)^2$ is frequently omitted by the computer of orbits for it gives very little definite information. . . . A glance at the numbers recorded in Table II [appendix] shows at once that no conclusions could be based upon any means that might be taken." This is a hard saying, for it implies that from all the work spent in deriving the orbits of single-spectrum stars no information bearing on the masses of the systems can be deduced.

This familiar mass function, which can be computed very simply and with considerable accuracy in general, conveys all that is directly known about the masses in a system when the second spectrum is absent. It is true that its values show a disconcertingly wide dispersion. The distribution is not normal, and the arithmetic mean for any class of star would have little value. In such circumstances the median seems more appropriate. It is not the probable value, but it may be regarded as the typical value in the sense that any observed value of the class considered is equally likely to exceed or fall short of it.

2. When the median value of the mass function has been ascertained and adopted, a corresponding value of $m \sin^3 i$ is required. In the first instance, in order to obtain a provisional result, the mean values found by Dr. Aitken from the double-spectrum binaries have been accepted. This makes it possible to form the quotient $m(m+m_1)^2/m_1^3$, and hence to infer the mass ratio m_1/m.

The first half of Table I is reproduced from Dr. Aitken's book.† The figure $m \sin^3 i = 0{\cdot}70$ for the K-type stars is an addition to be explained later. In the second half is given the number of stars N in each class, the median value of the mass function, the quotient formed with the mean $m \sin^3 i$ in the third column, the ratio m_1/m inferred and the product $m_1 \sin^3 i$.

So long as the attention is confined to the systems showing a double spectrum, the components are nearly equally bright and their masses are not very unequal. High mass ratios are to be expected and are found. When the second spectrum is absent the contrast in brightness is greater and the mass ratios are correspondingly smaller. They appear to rise in the classes G5–K, and the reason seems plain. Here, with a single exception, the second spectrum has not been distinguished, and all the mass ratios which occur are ranged on one side of the table. In the earlier classes the larger and smaller values are more or less effectively separated.

3. Table I presents merely a preliminary view of the kind of results

* 2nd edition, p. 223.

† p. 222.

TABLE I

Class	Double-spectrum Binaries (Aitken)				Single-spectrum Binaries				
	N	$m \sin^3 i$	$m_1 \sin^3 i$	m_1/m	N	$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	$\frac{m(m+m_1)^2}{m_1^3}$	$\frac{m_1}{m}$	$m_1 \sin^3 i$
O	1	32·2	24·3	0·75	5	1·34	24	0·44	14·2
Bo–B2	10	10·79	8·09	0·75	11	0·36	30	0·40	4·3
B3–B5	16	7·44	4·77	0·64	30	0·047	158	0·21	1·56
B8–A3	27	2·08	1·61	0·77	60	0·043	48	0·33	0·69
A5–F4	11	1·33	1·04	0·78	15	0·033	40	0·36	0·48
F5–G2	18	1·26	1·16	0·92	36	0·028	45	0·34	0·43
G5–G6	1	0·80	0·70	0·88	14	0·101	8	0·72	0·58
K	...	(0·70)	...	...	29	0·064	(11)	(0·62)	(0·43)
	84				200				

to be expected. It is based on the median values of the mass function for the single-spectrum binaries, and on the mean values of $m \sin^3 i$ for the double-spectrum binaries. To be consistent, median values should have been used in both cases. In forming the means certain exceptional stars have been omitted for obvious reasons, and it may be worth while to consider the actual material.

In the class Bo–B2 the figures for $m \sin^3 i$ are :

11·2, 13·0, 13·85, 17·57, 18·5, 75·6
10·3, 9·6, 5·53, 5·42, 2·90

when arranged in ascending and descending order from the median. The eclipsing binary Y Cygni is not included. Apart from the extreme pair the distribution is symmetrical. The median 11·2 may be a little low, but may be taken as the representative figure, the material being scanty.

In the class B3–B5 the figures similarly arranged are:

6·7, 7·0, 7·1, 7·5, 8·2, 11·4, 18·88, 20·8, 113·2
6·2, 6·01, 5·3, 5·25, 4·86, 4·8, 1·79, 1·5, 0·87

Symmetry is limited to the first five pairs, and a satisfactory mean can hardly be formed, even when the super-giant is excluded. As a representative value 6·4 may be adopted.

In the class B8–A3 the corresponding figures are:

1·71, 1·72, 1·83, 1·87, 1·97, 2·08, 2·19, 2·2
1·70, 1·62, 1·6, 1·48, 1·39, 1·18, 0·96

2·27, 2·47, 2·5, 2·87, 3·50, 4·3, 13·35
0·95, 0·91, 0·63, 0·56, 0·52, 0·14, 0·123

Here the material is more ample, and the median 1·71 can be adopted.

The figures for the class A5–F4 are:

1·04, 1·11, 1·19, 1·35, 1·53, 4·62
0·97, 0·91, 0·88, 0·74, 0·52, 0·28

Here the median 1·01 is well supported except by the extreme pair, and their evidence cannot be regarded as strong. But this value would upset the sequence, and make the masses in this class less than in the following class. As a matter of fact, there is no evidence, either from spectroscopic binaries or from eclipsing binaries, that the masses in these two classes are of a different order of magnitude. The two may therefore be combined, though the stars of type A5 may be fairly matched by stars in the preceding class, and the division ought perhaps to be made differently.

In the combined class A5–G2 the figures run:

1·19,	1·19,	1·26,	1·27,	1·30,	1·35,	1·36,	1·40,	1·46
	1·16,	1·12,	1·11,	1·05,	1·04,	1·01,	0·97,	0·94
	1·47,	1·48,	1·5,	1·53,	1·70,	1·79,	2·34,	4·62
	0·91,	0·88,	0·74,	0·67,	0·65,	0·52,	0·46,	0·28

and the median 1·19 can be accepted as the representative value of $m \sin^3 i$.

The effect of these changes on the results obtained in Table I are as follows :—

	B0–B2	B3–B5	B8–A3	A5–G2
$m \sin^3 i$	11·2	6·4	1·71	1·19
$m_1^3 \sin^3 i/(m+m_1)^2$	0·36	0·047	0·043	0·030
m_1/m	0·397	0·221	0·360	0·360
$m_1 \sin^3 i$	4·45	1·41	0·62	0·43

The alterations are seen to be unimportant.

4. Outside the range B0–G2 spectroscopic binaries with double spectrum are too few to give any trustworthy guide to the mass of the typical primary.

From Table I it is only possible to guess that the typical mass ratio for stars of type K should be 0·5, 0·6 or even greater. Now for two visual binaries there are the determinations :

70 Oph.	K0	$m = 0{\cdot}80$	$m_1 = 0{\cdot}72$
Melb. 4, AB	K2	0·67	0·51

These combined with seven other binaries of the same type, for which the parallaxes are known, give $m + m_1 = 1{\cdot}81$ in the mean. With

$$m_1^3 \sin^3 i/(m+m_1)^2 = 0{\cdot}064,$$

simple calculation gives

m_1/m	0·5	0·6	0·62
$m(m+m_1)^2/m_1^3$	18	11·9	11·0
$m \sin^3 i$	1·15	0·76	0·70
$m_1 \sin^3 i$	0·58	0·46	0·43

The statistical value of $\sin^3 i$, with due regard to the effect of natural selection, being about 2/3, it seems reasonable to take the third set of figures as fairly

typical. Accordingly, $m \sin^3 i = 0{\cdot}7$ will be adopted as a typical value for stars of type K.

For the class G5–G6, $m \sin^3 i = 1{\cdot}0$ will be adopted as a rough interpolation between the preceding and following classes.

The star 29 Can. Maj. (Oe) gives $m \sin^3 i = 32{\cdot}2$ directly. The slightly higher figure 36 is adopted in order to bring all the systems of this early class within the mass ratio 1.

5. The next point to be considered is the possibility of extending to individual systems the method already applied to groups.

Let $\mu = m \sin^3 i$ be the value adopted for the group, and let $f = m_1/m$ be the mass ratio in a particular system. This ratio is determined by an equation of the form

$$\frac{m(m+m_1)^2}{m_1{}^3} = \frac{(1+f)^2}{f^3} = \frac{\mu}{c}, \tag{1}$$

where c is the mass function. Hence corresponding errors are connected by the relation

$$\frac{\delta\mu}{\mu} = \frac{2\delta f}{1+f} - \frac{3\delta f}{f} = -\frac{3+f}{1+f} \cdot \frac{\delta f}{f}$$

or

$$\frac{\delta f}{f} = -\frac{1+f}{3+f} \cdot \frac{\delta\mu}{\mu}. \tag{2}$$

The reduced mass of the companion is $m_1 \sin^3 i = f\mu$. Its proportional error is therefore given by

$$\frac{\delta(f\mu)}{f\mu} = \frac{\delta f}{f} + \frac{\delta\mu}{\mu} = \frac{2}{3+f} \cdot \frac{\delta\mu}{\mu}. \tag{3}$$

When the mass ratio is known, the mean separation of the components (reduced by the factor $\sin i$) becomes known. It is

$$D = (a + a_1) \sin i = (1 + f^{-1}) a \sin i.$$

Thus

$$\frac{\delta D}{D} = \frac{\delta f}{1+f} - \frac{\delta f}{f} = \frac{1}{3+f} \cdot \frac{\delta\mu}{\mu}. \tag{4}$$

The detrimental effect of an error in the assumed value of μ on the scale of the orbit inferred is therefore greatly reduced.

6. The conditions arising from the nature of the mass function may also be considered graphically. The function determined is

$$\frac{m_1{}^3}{(m+m_1)^2} \sin^3 i = c.$$

Let

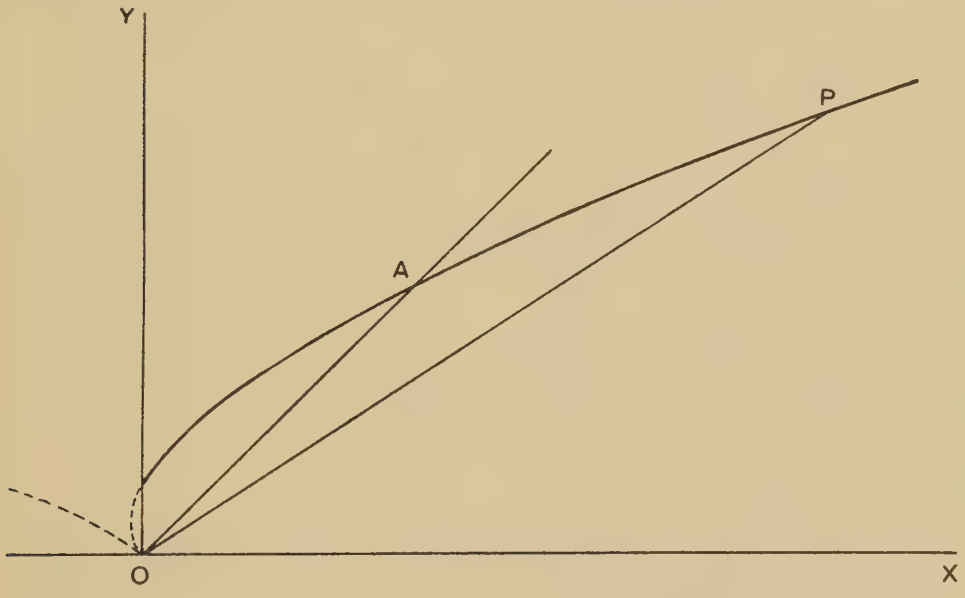
$$cx = m \sin^3 i, \qquad cy = m_1 \sin^3 i.$$

Then

$$y^3=(x+y)^2 \tag{5}$$

is a simple cubic curve with two infinite branches from a cusp at the origin. The tangent at the cusp is $x+y=0$, and since the curve does not cross this line at any other point the left-hand branch through $(-2, 1)$ is irrelevant. The right-hand branch also only becomes relevant when it enters the first quadrant at the point $(0, 1)$. From this point to $A(4, 4)$ the mass ratio $y/x > 1$; afterwards this ratio, measured by the slope of the radius vector OP, continues to decrease steadily (see diagram).

The assumption of the value of $m \sin^3 i = cx$ makes it possible to deduce the values of y and y/x from the curve, and hence of $m_1 \sin^3 i$ and m_1/m.



If the value of x is too large, that of y will also be too large and that of the mass ratio will be too small. But the shape of the curve makes it geometrically evident that the errors in these deduced quantities will be on a smaller scale than the error made in x. In particular, the error in the mass ratio will be relatively small when a small value of c makes x large.

7. For the simple calculation involved it is natural to write (1) in the form

$$\frac{f}{(1+f)^{\frac{2}{3}}}=\frac{c^{\frac{1}{3}}}{\mu^{\frac{1}{3}}}, \tag{6}$$

where c is the mass function. The mass ratio f in general assumes values from 1 downwards ; very rarely it may exceed unity, but not to a marked extent. Table II amply covers all requirements, and makes it easy to infer the value of f when c and μ are given.

For this purpose it is not the mass function in its accepted form that is wanted, but the cube root of this function. For every reason of

convenience it is suggested that the function to be calculated and tabulated should be

$$c^{\frac{1}{3}} = \frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$$
$$= [7{\cdot}67214] K P^{\frac{1}{3}} (1-e^2)^{\frac{1}{2}} \qquad (7)$$

in a notation which needs no explanation. The values may be recorded uniformly to three decimal places.

The fact that it is the cube root, $\mu^{\frac{1}{3}} = m^{\frac{1}{3}} \sin i$, which occurs in (6), makes it unnecessary to require a high degree of accuracy in the value of μ.

TABLE II

$f(1+f)^{-\frac{2}{3}}$

f	0	1	2	3	4	5	6	7	8	9
·0	·000	·010	·020	·029	·039	·048	·058	·067	·076	·085
·1	·094	·103	·111	·120	·128	·137	·145	·153	·161	·169
·2	·177	·185	·193	·200	·208	·215	·223	·230	·238	·245
·3	·252	·259	·266	·273	·280	·287	·293	·300	·307	·313
·4	·320	·326	·332	·339	·345	·351	·357	·364	·370	·376
·5	·382	·387	·393	·399	·405	·411	·416	·422	·427	·433
·6	·439	·444	·449	·455	·460	·465	·471	·476	·481	·486
·7	·491	·496	·502	·507	·512	·517	·521	·526	·531	·536
·8	·541	·545	·550	·555	·560	·564	·569	·573	·578	·582
·9	·587	·591	·596	·600	·604	·609	·613	·617	·621	·626

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1	0·630	·671	·709	·746	·781	·814	·846	·877	·906	·934
2	0·961	·988	*·013	·037	·061	·084	·107	·129	·150	·170
3	1·190	·210	·229	·248	·266	·284	·301	·319	·336	·352
4	1·368	·384	·399	·415	·430	·444	·459	·473	·487	·502

8. The circumstances, when normal, are thus favourable to the determination of the mass ratios, and even of the values of $m_1 \sin^3 i$, with fair accuracy. It is now necessary to consider the effect of less favourable conditions.

Abnormally large values of $m \sin^3 i$ clearly require m itself to be large. That means that the primary is a giant in its class. In this case the mass ratio inferred is too high and may exceed unity. Now the mass function is

$$c = \frac{m_1^3 \sin^3 i}{(m+m_1)^2} = (m+m_1) \sin^3 i \cdot \frac{f^3}{(1+f)^3},$$

so that

$$(m+m_1) \sin^3 i = c(1+f^{-1})^3 > 8c$$

and

$$(a+a_1) \sin i = a \sin i (1+f^{-1}) > 2a \sin i,$$

when it is assumed that $f < 1$. In those cases where it is found that $f > 1$ appreciably, this lower limit of $(a + a_1) \sin i$ has been given. Such systems are much more plausibly interpreted as exceptionally massive. They are the following :—

		$8c$			$8c$
ν Gem	B5	73·9	π Cep	G5	5·0
μ Sgr	B8p	32·0	Boss 5192	G5	4·1
υ Sgr	B8p	12·7	A Car	G5	2·5
Boss 809	Ao	8·2	ξ Pav	K2	9·5
β Cap	Go	9·4	Boss 1082	Ko	2·2

These figures represent lower limits which may be largely exceeded.

9. Abnormally low values of $m \sin^3 i$ will naturally occur owing to low inclinations of the orbits. Whereas there are independent means by which giant stars may be recognised, the inclinations remain undetermined. Hence it may seem that the present line of approach offers little prospect of success. Fortunately the effect of selection is to make high inclinations the rule, low inclinations the exception. The orbits to be first investigated are naturally those which give large variations in the radial velocity. Later, stars which show a smaller range of velocity will be included, and the problem will become more difficult. The present time is favourable because the material collected is now fairly ample, and it is not yet likely to include many orbits with low values of i.

When the systems are arranged in descending order of m_1/m, some will be placed too high in the list owing to exceedingly massive primaries, while others will be placed too low because the inclinations of the orbits are small. These opposite tendencies must be borne in mind. How far they are operative must be a matter for careful judgment.

10. Preliminary trials made on parts of the material having given encouraging results, the method has been extended to all the single-spectrum binaries of Table II in the appendix to Dr. Aitken's book. The outcome is given in tabular form at the end of this paper, for its general nature can scarcely be conveyed in a more summary way. The headings of the columns are sufficiently descriptive. The hypothetical elements are the mass ratio m_1/m, the reduced mass of the companion $m_1 \sin^3 i$ and the reduced separation $(a + a_1) \sin i$ of the components. The eccentricity e of the orbit is also added.

The unit of mass is that of the Sun. The unit of distance, in which $(a + a_1) \sin i$ is expressed, is 10^6 km.

In the classes Oe, Bo–B2 and B3–B5 the results obtained for the double-spectrum binaries are incorporated, being distinguished by the absence of a mass function and the unbracketed value of $m \sin^3 i$. Here, over a certain range of mass ratios, the hypothetical values can be confronted with those directly determined. In the later classes the figures for the double-spectrum binaries are not reproduced. In the classes B8–A3 and A5–G2 the mass ratios have been fairly successfully separated and there is little overlap between the two kinds of system. In the classes G5–G6 and Ko–K5 the single-spectrum binaries stand almost alone and cover the field.

11. Some comparison between the mass ratios in systems giving double and single spectra is exhibited in Table III. In addition to the numbers of binaries distributed according to class, nature of spectrum and mass ratio above and below the limit 0·8, the last column gives the proportion above this limit in each class irrespective of the resolution of the spectrum.

TABLE III

Class	Double Spectrum		Single Spectrum		Per cent.
	$\frac{m_1}{m} > 0{\cdot}8$	$\frac{m_1}{m} < 0{\cdot}8$	$\frac{m_1}{m} > 0{\cdot}8$	$\frac{m_1}{m} < 0{\cdot}8$	$\frac{m_1}{m} > 0{\cdot}8$
B0–B2	6	6	1	10	30
B3–B5	5	13	2	27	15
B8–A3	18	11	5	57	25
A5–G2	29	5	5	50	38
G5–K5	1	1	11	32	27
All	59	36	24	176	
Percentage of total (295)	20	12	8	60	28

In the class G5–K5 the spectrum is clearly not favourable to this resolution and systems with high mass ratio must be found, if at all, among the single-spectrum binaries. The general average, 28 per cent., is in fact maintained.

In the earlier classes, where the detection of the secondary spectrum seems to be easier, the higher mass ratios are chiefly to be found among the double-spectrum binaries. That the number of these ratios over 0·8 occurring among the single-spectrum binaries is small is a most important fact. For it seems to dispose conclusively of the fear that the ratio would be raised unduly by primaries of abnormal mass. Had this effect been operative, a far larger number of stars would have tended to encroach on the region which is mainly the province of the double-spectrum binaries.

12. The opposite tendency for the hypothetical mass ratios to be smaller than the true values cannot be tested so easily. Corresponding to the ratios assigned, values of the reduced mean separations, $(a+a_1)\sin i$, are now estimated for the first time. These are interesting as guides to the scale of the relative orbits. But they have another use. As the mass ratio descends to smaller values, the factor $(1+f^{-1})$ increases. Unduly small values of f will lead to exaggerated values of the separations, and to a more marked extent towards the end of the table. Now no such systematic tendency is apparent. On the contrary, there are not wanting cases which suggest that among the smaller mass ratios even smaller values are needed to avoid orbits below any reasonable limit of size.

While, however, the accuracy of individual figures cannot be expected, statistical deductions can be drawn from them. One example relating to the eccentricities of the orbits may be given. The combined class B8–G2 seems to afford tolerably homogeneous material. For different sizes of orbit the mean eccentricities are as follows :—

$(a+a_1)\sin i$	<10	10–20	20–50	50–100	>100
N	38	36	16	12	14
e	0·042	·171	·255	·292	·370

Of these 116 single-spectrum binaries, one-third form the class with the smallest orbits, and these the least eccentric. Few of the eccentricities exceed 0·1, and none exceeds 0·152 in this class. But one star has been arbitrarily excluded. It is the eclipsing binary U Cephei with the eccentricity 0·474. There can be little doubt that the scale of this orbit is much larger, and the mass ratio much smaller, than the figures here suggest. In other words, this star is a giant. That its exceptional position should be thus marked suggests that the other figures are not far from the truth. It is the exception which proves the rule in this case.

13. According to the evidence of the double-spectrum binaries the mass of the fainter star cannot greatly exceed the mass of the brighter, and is almost always the smaller of the two. Apparent exceptions to this rule among the single-spectrum binaries are capable of an alternative explanation (§ 8).

To the mass of the companion no lower limit can be assigned, for bodies of a planetary order are clearly possible. But the question remains, What is the lower limit in those systems which have been actually investigated? In these a measurable range of variation in the radial velocity requires the mass ratio to reach a certain value. The smallest masses of the fainter component are found in the classes B8–A3 and A5–G2. The stars which come into consideration are the following :—

		$\frac{m_1}{m}$	$m_1 \sin^3 i$			$\frac{m_1}{m}$	$m_1 \sin^3 i$
Boss 5671	A3	·058	0·10	ζ Tr A	Go	·079	0·09
$+47°$ 2227	Ao	·050	·09	α Tri	F5	·066	·08
β Ori	B8p	·040	·07	Boss 3511	Fo	·057	·07
γ UMi	A2	·026	·04	Boss 3544	F5	·057	·07
σ Cyg	Aop	·016	0·03	ξ UMa fr	Go	·037	0·04

It will be recalled that the smallest stellar mass determined is that of the fainter component of Krueger 60, about 0·16. The factor $\mathrm{cosec}^3 i$ need not be impossibly large to raise the reduced masses above to the same order of magnitude. Moreover, some of these stars are certainly giants, and in their case the values of $m_1 \sin^3 i$ are underestimated. On the whole there seems to be no evidence that in any of the systems so far investigated the smaller component is below the level of an ordinary stellar body.

When the possibility of raising a value of $m_1 \sin^3 i$ is contemplated, this implies a corresponding decrease in the value of m_1/m. The range of variation attributed to the mass ratio, so far from being diminished, is therefore increased. There is thus no reason to doubt that ratios approaching 1/100 have been actually reached in the systems observed. But ratios below 1/30 are very exceptional, and even this figure has not been reached in the earlier or the later classes. The range from 1 to 1/10 seems to have been explored in a fairly representative way.

14. Over this range a total of 269 stars showing either double or single spectra are distributed thus :

m_1/m	0·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
No.	29	27	32	27	29	30	25	34	36	

The distribution is therefore remarkably uniform. There may be some slight preference of the orbits of double-spectrum binaries for determination. Apart from this, the uniformity extends to the separate classes, so far as small numbers make it possible to judge. Hence it applies to the masses of the fainter components when the brighter and more massive components have been classified according to their spectral types. In view of the spurious dispersion to which these hypothetical masses are liable in both directions, to assert that their distribution is in fact uniform would be too positive a statement. But it does seem safe to say that evidence to the contrary is conspicuously absent. Had there been a marked maximum frequency or mode in the true distribution, it is strange that all trace of it should have been lost. It seems possible that the association of the masses in a spectroscopic binary is at random. Thus the masses of the companions associated with the K-type stars seem to resemble closely in their general run the corresponding figures in the lower half of the class B8–A3. There is no reason why the same K-type stars should not have been combined with the companions in the upper half of the same list. In that case the K-type star would play the subordinate part, and the binary would have been classified differently.

15. The primary object in view has been to lay stress on the importance of the mass function $m_1{}^3 \sin^3 i/(m+m_1)^2$ obtained for a spectroscopic binary. At first sight it is a function of three variables m, m_1 and i. But it may also be regarded as a function of the two variables, $m \sin^3 i$ and $m_1 \sin^3 i$, which are separately determined when the two spectra are visible. A certain inconvenience in the arithmetical magnitude of the function can be obviated by substituting its cube root.

In making a general review of the material now available, the evidence of the spectroscopic binaries themselves has been used exclusively, with as little appeal to other sources of information as possible. This course seems reasonable, since no systematic examination of the mass relations appears to have been made hitherto. The subject needs, and will no doubt receive in future, a more critical and detailed discussion.

Similarly no attempt has been made to give particular consideration to individual systems. Some of these are of special interest and they receive scant justice in a summary treatment. The number of single-spectrum binaries with orbits determined is twice the number of double-spectrum binaries studied in an equal degree. The material is thus more ample, and in a sense less restricted, though at the same time less specific with regard to the masses of the systems.

The contents of the concluding Table IV are for the most part sufficiently indicated by the headings of the columns, but further explanations have been given in § 10.

TABLE IV

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m \sin^3 i$	$m_1 \sin^3 i$	$(a+a_1) \sin i$	e
			Oe : $m \sin^3 i = 36$				
Boss 450	Oe5	2·068	0·99	(36·0)	35·7	63·7	0·065
29 CMa	Oe	...	·75	32·2	24·3	504·7	·156
τ CMa	Oe5	1·230	·49	(36·0)	17·5	317·5	·360
ι Ori	Oe5	1·102	·42	,,	15·2	102·8	·742
+44° 3639	Oe5	0·721	·25	,,	9·1	138·6	·099
Boss 5565	Oe5	0·537	0·18	,,	5·9	26·9	0·188
			Bo–B2 : $m \sin^3 i = 11·2$				
Y Cyg	B2	...	1·01	17·4	17·6	19·9	0·130
η Ori	B1	...	0·95	11·2	10·6	32·7	·016
Boss 46	Bo	...	·93	17·57	16·37	21·9	·037
+35° 3970	Bo	...	·90	13·85	12·90	36·0	·264
+6° 1309	Bo*p*	...	·84	75·6	63·3	89·7	·035
+45° 3139	B1	...	·81	2·90	2·35	10·5	·098
θ^2 Ori br.	B1	1·215	·80	(11·2)	9·0	60·5	·270
ψ Ori	B2	...	·76	5·53	4·19	11·6	·065
o Per	B1	...	·70	5·42	3·79	16·5	·000
Boss 6142	Bo	...	·69	18·5	12·7	51·9	·105
β' Sco	B1	...	·64	13·0	8·3	29·2	·270
Boss 5070	B2	1·007	·62	(11·2)	7·0	41·2	·199
α Vir	B2	...	·60	9·6	5·8	18·3	·100
Boss 1353	Bo	0·886	·52	(11·2)	5·9	1589	·220
δ Ori	Bo	0·849	·50	,,	5·6	24·0	·107
χ Aur	B1	0·824	·48	,,	5·3	564·5	·171
VV Ori	B2	0·710	·39	,,	4·4	9·6	·000
Boss 1464	B2	...	·38	10·3	3·9	28·0	·250
γ Ori	B2	0·652	·36	(11·2)	4·0	187·3	·599
ϕ Per	Bo*p*	0·573	·31	,,	3·4	191·2	·428
α^2 Cru	B1	0·138	·064	,,	0·7	7·2	·060
ν Cen	B2	0·134	·062	,,	0·7	12·8	·000
α' Cru	B1	0·112	0·052	,,	0·6	7·6	0·322
			B3–B5 : $m \sin^3 i = 6·4$				
ν Gem	B5	2·10	2·71	(6·4)	(17·3)	> 2834	0·200
Boss 5375	B3	...	0·93	1·79	1·67	8·8	·137
+33° 785	B3	...	·88	4·86	4·29	9·7	·051
+64° 1717	B3	...	·86	11·4	9·8	11·9	·034
κ Vel	B3	1·05	·85	(6·4)	5·5	158·9	·190
σ Aql	B3	...	·83	5·3	4·4	9·7	·000
Boss 5764	B5	...	·82	0·87	0·71	6·4	·015
TT Aur	B5	...	·79	6·7	5·3	8·1	·000
π And	B3	0·962	·76	(6·4)	4·8	179·5	·573
ν And	B3	...	·73	1·50	1·10	10·6	·000
−3° 1413	B5	...	·66	6·2	4·1	7·1	·036
Boss 5629	B3	...	·65	20·8	13·6	63·6	·224
Boss 5834	B3	...	·64	6·01	3·87	30·9	·253
Boss 1399	B3	0·847	·63	(6·4)	4·1	57·7	·765
−10° 4926	B5	...	·62	7·10	4·43	10·0	·033
+58° 2546	B3	...	·60	4·8	2·9	21·7	·376
+47° 3692	B3	0·777	0·56	(6·4)	3·6	10·6	0·000

TABLE IV—*continued*

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m \sin^3 i$	$m_1 \sin^3 i$	$(a+a_1) \sin i$	e
ϵ Vol	B5	0·759	0·55	(6·4)	3·5	36·8	0·000
Boss 3138	B3	...	·54	8·2	4·4	12·6	·057
+7° 676	B5	...	·53	7·0	3·7	10·9	·076
+59° 609	B5	...	·49	18·88	9·17	20·0	·102
δ' Lyr	B3	0·676	·47	(6·4)	3·0	122·7	·280
Boss 1216	B3	0·654	·45	,,	2·9	92·7	·189
Z Vul	B3	...	·45	5·25	2·37	10·5	·000
+20° 4022	B3	0·643	·44	(6·4)	2·8	38·9	·159
+45° 4147	B5	0·599	·41	,,	2·6	13·6	·233
+57° 28	B5	...	·40	113·2	44·9	231·3	·033
u Her	B3	...	·39	7·5	2·9	9·9	·053
AR Cas	B3	0·493	·32	(6·4)	2·0	20·3	·224
Boss 1159	B3	0·421	·27	,,	1·7	14·0	·000
λ Tau	B3	0·418	·26	,,	1·7	14·6	·060
θ^2 Cru	B3	0·363	·22	,,	1·4	13·2	·000
ζ Tau	B3p	0·357	·22	,,	1·4	155·3	·180
ξ^2 Cen	B3	0·336	·20	,,	1·3	21·6	·353
Boss 1388	B3	0·304	·18	,,	1·2	52·5	·550
+13° 4098	B3	0·301	·18	,,	1·2	10·6	·056
Boss 1147	B3	0·258	·15	,,	1·0	25·7	·027
Boss 5913	B3	0·240	·14	,,	0·90	29·5	·100
Boss 1107	B5	0·238	·14	,,	·90	7·4	·087
H Vel	B5	0·201	·12	,,	·74	5·1	·131
Boss 2473	B3	0·188	·11	,,	·69	20·1	·180
Boss 3586	B5	0·187	·11	,,	·69	20·4	·230
π Ari	B5	0·183	·11	,,	·68	13·7	·042
θ' Sgr	B3	0·097	·054	,,	·35	9·0	·000
κ Dra	B5p	0·084	·047	,,	·30	5·1	·186
α Pav	B3	0·079	·044	,,	·28	27·8	·010
Boss 4870	B3	0·058	0·032	,,	0·20	5·5	0·015

B8–A3 : $m \sin^3 i = 1{\cdot}71$

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m_1 \sin^3 i$	$(a+a_1) \sin i$	e
μ Sgr	B8p	1·587	3·75	(6·4)	> 296·2	0·447
υ Sgr	B8p	1·165	2·05	(3·5)	> 182·0	·087
Boss 809	A0	1·007	1·59	(2·7)	> 29·9	·082
α Dra	A0p	0·747	0·99	1·69	60·8	·384
RR Lyn	A3	·677	·85	1·46	19·8	·081
U Cep	A0	·617	·75	1·28	7·8	·474
+25° 4299	A2	·596	·71	1·22	20·8	·290
κ Cnc	B8	·585	·70	1·19	14·3	·149
α And	A0p	·565	·66	1·13	87·3	·525
21 Cas	A2	·556	·65	1·11	11·2	·000
ω Cas	B8	·547	·63	1·08	70·1	·300
+18° 4794	A3	·544	·63	1·08	32·0	·000
Boss 1076	A3	·522	·60	1·02	9·6	·000
+32° 840	A3	·521	·59	1·02	15·0	·033
+34° 610	A2	·517	·59	1·01	12·8	·040
Boss 1452	A0	·511	·58	0·99	8·3	·030
Boss 4008	B8	0·510	0·58	0·99	46·9	0·773

TABLE IV—*continued*

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m_1 \sin^3 i$	$(a+a_1)\sin i$	e
Boss 5996	A3	0·510	0·58	0·99	8·8	0·036
95 Leo	A2	·508	·58	·98	14·4	·020
η Vir	Ao	·501	·56	·97	71·3	·400
Boss 4876	A3	·482	·54	·92	5·4	·008
δ Lib	Ao	·476	·53	·90	7·1	·054
+12° 3241	B9	·446	·49	·83	14·3	·069
+27° 2138	A3	·446	·48	·83	20·4	·060
+33° 2574	A2	·420	·45	·77	9·2	·079
α CrB	Ao	·392	·41	·71	26·2	·387
ζ' Lyr	A3	·392	·41	·71	10·4	·000
Boss 4507	Ao	·391	·41	·70	8·0	·060
Boss 613	A2	·353	·36	·62	7·2	·014
RZ Cas	Ao	·345	·35	·60	4·4	·052
Boss 1018	A2	·345	·35	·60	16·0	·160
TX UMa	B8	·339	·34	·59	81·7	·205
+12° 3161	Ao	·335	·34	·58	31·5	·427
Boss 2754	Ao	·335	·34	·58	19·8	·381
Boss 4779	B9	·330	·33	·57	6·1	·116
+11° 2662	A3	·307	·31	·52	14·2	·199
Boss 82	A2	·307	·31	·52	9·5	·152
+8° 3134	A2	·284	·28	·48	16·4	·376
ι Del	A2	·271	·27	·45	18·6	·252
ϕ Aql	A2	·268	·26	·45	8·4	·055
Boss 1161	A2	·262	·25	·43	9·3	·013
Boss 4669	A2	·252	·24	·42	17·0	·468
μ Ori	A2	·238	·23	·39	10·7	·000
Boss 159	B8	·230	·22	·37	39·1	·405
Boss 6070	Ao	·222	·21	·36	23·6	·037
α' Gem	Ao	·213	·20	·34	7·6	·010
+34° 2971	B9	·213	·20	·34	12·1	·031
Boss 986	A2	·210	·20	·34	4·9	·017
+17° 2402	A2	·208	·20	·34	7·2	·018
Boss 5442	Ao	·182	·17	·29	8·2	·000
ϵ UMa	Ao*p*	·180	·17	·28	487·3	·310
Boss 2035	A2*p*	·177	·16	·28	97·4	·198
Boss 1244	A2	·157	·14	·24	9·0	·033
Boss 4864	B8	·141	·13	·22	4·4	·000
+57° 651	Ao	·119	·11	·18	14·8	·227
α^2 Gem	Ao	·114	·10	·17	16·2	·503
Boss 4177	Ao	·112	·10	·17	10·5	·511
Boss 5671	A3	·067	·058	·10	13·6	·004
+47° 2227	Ao	·058	·050	·09	3·3	·000
β Ori	B8*p*	·047	·040	·07	28·8	·296
γ UMi	A2	·031	·026	·04	1·4	·090
σ Cyg	Ao*p*	0·019	0·016	0·03	17·5	0·400
		A5–G2 : $m \sin^3 i = 1·19$				
β Cap	Go, Ao	1·056	2·13	(2·53)	> 765	0·437
β Sct	Go	0·691	1·05	1·25	347·5	·350
μ Per	Go	0·633	0·92	1·10	166·8	·062
η Peg	Go	0·616	0·89	1·06	335·5	0·155

TABLE IV—*continued*

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m_1 \sin^3 i$	$(a+a_1)\sin i$	e
ι TrA	Fo	0·597	0·85	1·01	44·4	0·280
Boss 1492	A5	·553	·76	0·91	16·6	·208
α^2 Her	F9	·519	·70	·83	5·1	·028
Boss 3098	F8	·518	·70	·83	63·8	·008
ι Peg	F5	·489	·64	·76	17·2	·008
+75° 342	A5	·482	·63	·75	9·6	·109
+82° 325	Go	·481	·63	·75	25·8	·282
χ Dra	F8	·480	·63	·75	156·1	·419
Boss 4797	Go, A3	·467	·60	·72	142·3	·120
θ^2 Tau	Fo	·462	·59	·71	100·4	·717
TW Dra	A5	·436	·55	·65	7·1	·054
+39° 4529	F8	·433	·54	·65	7·9	·022
+25° 1594	Go	·413	·52	·61	36·4	·080
Boss 2463	F8	·412	·51	·61	22·8	·090
+22° 2731	F5	·411	·51	·61	77·0	·101
Boss 523	F8	·371	·45	·53	72·3	·445
κ Peg	F5	·354	·42	·50	11·5	·034
m Vel	F8	·351	·42	·50	61·5	·050
β Ari	A5	·347	·41	·49	78·4	·880
β Tri	A5	·341	·40	·48	34·6	·456
α UMi	F8	·327	·38	·45	1686	·630
Boss 3182	A5	·322	·38	·45	4·0	·000
η Boo	Go	·315	·37	·44	215·5	·236
Boss 4322	F5	·311	·36	·43	47·7	·128
δ Cap	A5	·311	·36	·43	3·5	·019
ω Dra	F5	·297	·34	·40	10·4	·011
ϵ Lib	Fo	·294	·34	·40	127·4	·680
+13° 3258	F2	·288	·33	·39	17·6	·308
ξ UMa, br	Go	·278	·32	·38	259·0	·531
Boss 1593	F5	·278	·31	·37	11·9	·019
ϵ Hya C	F5	·274	·31	·37	15·7	·620
OΣ 82, br	Go	·268	·30	·36	8·6	·060
+8° 2134	Go	·249	·28	·33	15·9	·100
Boss 2285	F5	·247	·27	·33	20·0	·276
+72° 227	A5	·241	·27	·32	8·7	·043
13 Cet	Go	·225	·25	·29	5·4	·022
Boss 593	F5	·224	·24	·29	15·4	·146
ρ Tuc	F5	·207	·22	·27	9·5	·024
Boss 2227	F5	·165	·17	·21	4·4	·051
θ Dra	F8	·160	·17	·20	68·8	·014
Boss 3644	F5	·159	·17	·20	6·3	·000
Boss 2665	F5	·150	·16	·19	14·4	·048
BDS 9818 br	F3	·147	·15	·18	19·1	·100
R CMa	Fo	·139	·14	·17	3·5	·138
ξ Ser	A5	·119	·12	·14	5·7	·000
δ Tri	Go	·089	·089	·11	14·7	·059
ζ TrA	Go	·079	·079	·09	18·0	·060
α Tri	F5	·067	·066	·08	4·6	·121
Boss 3511	Fo	·058	·057	·07	4·1	·054
Boss 3544	F5	·058	·057	·07	14·7	·247
ξ UMa fr	Go	0·038	0·037	0·04	7·7	0·000

TABLE IV—*continued*

G5–G6 : $m \sin^3 i = 1{\cdot}0$

Star	Type	$\frac{m_1 \sin i}{(m+m_1)^{\frac{2}{3}}}$	$\frac{m_1}{m}$	$m_1 \sin^3 i$	$(a+a_1) \sin i$	e
π Cep	G5	0·854	1·63	(1·63)	> 338	0·281
Boss 5192	G5	·800	1·46	(1·46)	> 186	·042
A Car	G5	·679	1·12	(1·12)	> 134	·000
Boss 1381	G5	·512	0·74	0·74	112·6	·510
ϵ UMi	G5	·511	·74	·74	40·8	·011
+55° 2215	G6	·496	·71	·71	79·5	·054
+16° 577	G5	·488	·69	·69	11·0	·040
m Vel	G5	·457	·63	·63	164·3	·019
δ Col	G5	·343	·44	·44	299·6	·695
+11° 2910	G5	·290	·36	·36	8·5	·089
Boss 5566	G5	·254	·30	·30	17·9	·195
Boss 2447	G5	·250	·30	·30	23·1	·206
Boss 1131	G5	·210	·24	·24	39·9	·391
Boss 1508	G5	0·114	0·12	0·12	13·8	0·206

Ko–K5 : $m \sin^3 i = 0{\cdot}7$

ξ Pav	K2	1·059	3·01	(2·11)	> 1052	0·264
Boss 1082	Ko	0·654	1·27	(0·89)	> 94	·019
o Dra	Ko	·568	1·02	0·71	> 56	·114
σ Pup	K5	·547	0·97	·68	131·7	·170
Boss 4629	K5	·543	·95	·67	198·4	·398
Boss 3180	K5	·510	·87	·61	191·1	·169
α Tuc	K2	·506	·86	·60	826·2	·385
Boss 3649	Ko	·479	·80	·56	206·0	·070
Boss 5900	Ko	·455	·74	·52	26·5	·000
Boss 2490	Ko	·437	·70	·49	293·4	·293
Boss 3547	Ko	·434	·70	·49	178·4	·134
σ Gem	Ko	·433	·69	·49	22·5	·022
γ Phe	K5	·429	·68	·48	103·6	·005
Boss 3673	Ko	·424	·67	·47	109·5	·540
α Phe	Ko	·400	·62	·43	756·0	·335
β Her	Ko	·373	·57	·40	166·6	·550
ι Gru	Ko	·359	·54	·38	165·5	·656
Boss 804	Ko	·351	·52	·37	291·1	·397
Σ 1561 fr	K4	·324	·47	·33	24·3	·350
ζ And	Ko	·318	·46	·32	20·0	·017
+43° 2135	K2	·316	·46	·32	16·2	·080
33 Psc	Ko	·311	·45	·31	51·3	·272
β Ret	Ko	·296	·42	·29	450·3	·210
Boss 5294	Ko	·269	·37	·26	100·5	·128
+27° 4642	Ko	·232	·31	·22	14·9	·059
λ Hya	Ko	·203	·27	·19	382·3	·138
τ Pup	Ko	·198	·26	·18	293·2	·088
ζ Cet	Ko	·149	·19	·13	382·2	·586
λ And	Ko	0·089	0·11	0·07	20·8	0·086

16. Two opposite views can be taken of a purely tentative investigation like the present. The material is strongly affected by the element of selection, and there may be wide systematic differences between the systems with single and double spectra. If such apprehensions are accepted *a priori* and pushed to their logical conclusion, the problem will be regarded as a hopeless one. All knowledge of the general character of the smaller mass ratios, to which the double-spectrum systems do not descend, will be inhibited.

But a different view has been taken here. While the difficulties and uncertainties of the problem are fully admitted, they have been deliberately ignored of set purpose in order to arrive at a plausible picture, which can be judged on its own merits.

The adopted values of $m \sin^3 i$ are admittedly rough estimates. In so far as they are based on double-spectrum binaries it has been suggested (without precise proof) that they are systematically too large. Unless they are very much too large for the present application, they will serve for the very modest and limited object in view. If, on the contrary, they were altogether excessive, the deduced mass ratios would be systematically too small, and evidence for this should appear in a discontinuity between the double- and single-spectrum systems. Now continuity in this respect is one of the more striking features emerging from the discussion, and it has been examined with some care. Further, the deduced values of $(a + a_1) \sin i$ should be systematically too large. They were, in fact, brought in to test this very point. Again, the evidence to all appearance fails to substantiate the suspicion that the results, such as they are, are invalidated by systematic tendencies.

But when standard values of $m \sin^3 i$ are assumed and applied to individual systems, it may well seem that rashness has outrun discretion. Again, in the last resort, the results must speak for themselves, and that is why they have been reproduced *in extenso*. Here the difficulty lies in the undoubted dispersion of the true values of $m \sin^3 i$, or rather of $m^{\frac{1}{3}} \sin i$. This is mainly due to the dispersion in the unknown inclinations i. But it is a reasonable hope that this dispersion is in fact largely limited by the selective element in the material. The need for high inclinations (or large K) when a double spectrum can be resolved, on which so much stress has been laid elsewhere, applies no less to the single-spectrum systems which first invite attention. This contention becomes precarious if pressed too far, but it may help to explain why the results in Table IV appear tolerably correct in broad outline though they must be faulty in detail.

It has been assumed that the mass m itself of a star within a limited range of spectral type tends closely to a typical value. The distinction between giants and dwarfs has been ignored. These must be separated when a more refined investigation is undertaken. When more material is available it should also be possible to introduce a more refined spectral classification.

No reference has been made to earlier work in this field. An account of the literature of the subject down to 1927 has been given by Beer.*

* *Veröff. der Universitätssternwarte zu Berlin-Babelsberg*, Band V, Heft 6 (1927).

Summary

1. The mass function determined from a single-spectrum binary may be regarded as a function of $m \sin^3 i$ and the mass ratio $f = m_1/m$. Typical values of $m \sin^3 i$ for different classes have been adopted, either from median values or otherwise (§§ 3, 4).

2. A simple method of deriving f is given, and the effects of errors in the adopted values of $m \sin^3 i$ are discussed (§§ 5–7).

3. Typical results are shown in Table I, based on mean values of $m \sin^3 i$ and median values of the mass function ; they are later adjusted to median values of $m \sin^3 i$ (§§ 2, 3).

4. The dispersion to which values of $m \sin^3 i$ are liable within each class is discussed (§§ 8, 9).

5. The method is applied to individual systems and the results are given in Table IV (§§ 10, 15).

6. The incidence of high and low mass ratios in the various classes is shown in Table III. The results are compared with those for the double-spectrum binaries and are shown to be complementary (§ 11).

7. The mass ratios now obtained make it possible to estimate the reduced separations $D = (a + a_1) \sin i$. These are useful as a check on systematic tendencies and are further confirmed for a large class of stars by the eccentricities of the orbits (§ 12).

8. The smallest masses for the companions are found in the class B8–G2. There is no clear evidence that bodies smaller than ordinary dwarf stars are involved (§ 13).

9. In each class the mass ratios seem to be uniformly distributed. This unexpected result is inherent in the run of the cube roots of the mass functions. No evidence of correlation has been found between the masses of the primary and the companion, apart from the accepted principle that the brighter component is also in general the more massive (§ 14).

OBSERVATIONS OF SS AURIGÆ (060547).

(First Series: 1910–1914.)

P. M. Ryves.

1. The observations of SS Aurigæ made at Zaragoza, Spain, during the years 1923–1932 (inclusive) have been communicated to the Society in two five-yearly instalments,* and another instalment will be ready, it is hoped, at the beginning of 1938.

There is an earlier series consisting of 322 observations made in the years 1910–1914, an abstract of which, referring to the positive observations only, was published in 1924.† Since, with the instruments employed, this *U Geminorum* type variable is invisible except during its irregularly periodic outbursts, the negative observations, which define the periods of quiescence, are much more numerous than the positive. Requests for the former having been received, it has been decided to put these on record together with the 67 positive observations which are now given in full. The present paper is thus uniform with those containing the later observations and constitutes the "first series." The longer preliminary explanation applies to the later series also.

During the interval between the first and second series, *i.e.* the years 1915–1922 (incl.) no observations were made here.

2. SS Aurigæ having been announced ‡ as a variable with maximum magnitude of $8^{m}\cdot5$, the 3-inch portable telescope (refractor) was chosen for the purpose of observation, it being more convenient to use this instrument; but later, when it became evident that maximum brightness was of the order of 11th magnitude only, the 5-inch refractor was generally preferred. No other instruments were used for the first series; any other telescopes used in later years will be mentioned in the notes to each series.

As regards the method of observation (and this applies equally to the later series), the step method of Herschel and Argelander has been employed, each comparison providing an independent determination of magnitude, the deduced magnitude in column 4 being the mean, sometimes the weighted mean, of the several estimates. In reducing, the estimates should *not* be treated as simple interpolations, as in the so-called "fractional method."

In the negative observations, comparison stars are always used in order to estimate the limiting magnitude.

3. Table I contains details of the comparison stars which are taken from the Harvard sequence.§ The designations are those of H. C. O. There exist two series of photometer measures of the brighter stars, one made with the 12-inch meridian photometer at Harvard, the other with a wedge photometer attached to the 26-inch refractor at the Leander McCormick

* *M.N.*, **88**, 607, 1928, and **94**, 246, 1934.
† *M.N.*, **84**, 624, 1924.
‡ *A.N.*, **174**, 363, and *H.B.*, 324.
§ *H.C.*, 138.

Observatory,* Virginia. Down to *q*, following the lettering, the values in column 4 are the means of these two series of measures (except *g*, which depends upon Harvard alone). The remainder are McCormick alone. The Adopted Magnitudes (column 5) are those of the McCormick revision, with the exception of *r*, to which I have assigned $12^m.30$ instead of the 12·37 of McCormick. The latter value is probably affected by an error in the original Harvard estimate, which has not been entirely eliminated. The spectrum is probably G. The reddish stars *l* and *o* have been avoided as far as possible, especially the latter, which is possibly variable to a small degree. The sequence in general is satisfactory, and is in good accordance with the observer's estimates.

TABLE I

Comparison Stars for SS Aurigæ. Co-ordinates :

The variable { R.A. (1855) $6^h\ 02^m\ 24^s$. Dec. (1855) +47° 46′.
R.A. (1900) $6^h\ 05^m\ 48^s.5$. Dec. (1900) +47° 34′·7.

Designation	Δ R.A.	Δ Dec.	Photometer Mag.	Adopted Mag.
	m s	′		
f	+0 23	−10·5	10·51	10·52
g	+0 19·5	− 3·4	*10·78*	10·77
k	+0 00·5	+ 2·0	10·93	10·82
l	+0 42·5	+ 3·5	10·92	11·00
m	−0 19	− 8·3	11·21	11·20
n	+1 16·5	+ 4·0	11·53	11·46
o	−0 27·5	− 5·9	11·37	11·51
p	−0 00·5	−13·6	11·70	11·78
r	−0 33·5	+ 2·4	12·05	*12·30*
q	−0 21	+ 0·5	12·46	12·46
s	−0 30	− 4·5	13·05	13·02
t	−0 21·5	− 1·6	13·11	13·21
u	−0 13	− 1·5	13·79	13·78
w	−0 09·5	+ 0·5	13·94	13·94
x	+0 06·5	± 0·0	14·64	14·54
y	+0 09	− 0·3	...	14·83
z	+0 06·5	− 0·4	...	15·2
a	−0 01·5	− 0·5	...	15·8

In Table II the day is astronomical, commencing at noon. In the column headed " comparisons," designation letters alone indicate that such comparison stars were used, the estimates not being recorded ; the limiting magnitude given in column 4 is, however, the resultant of the unrecorded estimates. More usually step intervals are given, negative observations being recorded in the form *t*2N (as in the 6th observation) signifying that the variable was invisible and more than two steps ($0^m.1$) fainter than *t*, the value in column 4 being the arithmetical result—in this case $13^m.3$. The fact that a certain comparison star was seen and the variable not seen cannot always be considered as conclusive evidence that the latter was

* *A.J.*, 779.

fainter than the former, and observations are occasionally recorded in the form 2*t*N, which gives a limiting magnitude two steps *brighter* than *t* (*e.g.* 1912 Sept. 17). A designation letter in brackets (*t*) indicates that such star was seen with difficulty, or " glimpsed."

The following arbitrary symbols have been used. It will be noted that they have been designed to occupy the smallest possible space. Intensities of moonlight, twilight, haze, etc. are not given here but can be obtained by reference to the original manuscript record.

Symbols and Abbreviations

W. = weight, 1 to 5.	j = unsteady (jumpy).
Sk. = sky conditions near the variable.	i = interference from clouds.
! = clear transparent sky.	(= moonlight, moderate or strong.
: = clear but less transparent.	t = twilight, moderate or strong.
; = haze or mist.	? = doubtful observation.
. = star low ; clear unless otherwise stated.	t″ = star (t) to right of variable.

A weight number in *italics* indicates that the 3-inch telescope was used, otherwise the 5-inch is understood.

The step or grade has been taken as $0^m.05$, which is very near to the value derived from this series of observations.

TABLE II

	G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1910	h					
Oct. 31	9·5	8976·40	*r*, *t*	< 13·2	4	!
Nov. 3	10·4	8979·43	*t*	< 13·2	4	!
5	8·1	8981·34	*r*, *q*	< 12·5	*2*	!
6	14·3±	8982·60	*r*, *q*	< 12·5	*2*	!
7	14·5	8983·60	*r*, *q*, *t*	< 12·6	*4*	!
8	10·0	8984·42	*t*2N	< 13·3	*3*	!
10	13·2	8986·55	*r*, *q*	< 12·5	*3*	!
11	13·6	8987·57	*r*, *q*	< 12·7	*4*	!
20	7·6	8996·32	*r*, *q*	< 12·5	3	!
22	8·1	8998·34	*r*, *q*	< 12·4	*3*	!
23	7·5	8999·31	*r*	< 12·3	*3*	!
24	9·5	9000·40	*r*	< 12·4	*3*	!
29	11·2	9005·47	*q*	< 12·3	*3*	!
30	9·5	9006·40	*r*?	< 12·2	*2*	!
Dec. 1	6·8	9007·28	*o*	< 11·8	*3*	!
2	8·8	9008·37	*o*	< 11·5	*1*	i
3	7·6	9009·32	*o*, *p*	< 11·7	*3*	i
4	14·7	9010·61	*q*oN	< 12·45	*4*	!
5	14·4	9011·60	*p*, *q*	< 12·3	*3*	;
6	10·1	9012·42	*q*2N	< 12·5	*3*	:
7	6·2	9013·26	*p*	< 12·0	*4*	:

TABLE II—*continued*

	G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1910	h					
Dec. 8	6·9	9014·29	*p*0N	< 11·8	3	(!
10	9·0	9016·37	*p*	< 12·0	3	(!
11	15·6	9017·65	*q*2N	< 12·55	4	!
12	6·8	9018·28	*m*4N	< 11·2	3	(:
13	7·2	9019·30	*k*4N	< 11·0	3	(:
14	13·8	9020·57	*m*0N	< 11·2	2	(:
17	9·7	9023·40	*m*0N	< 11·2	3	(:
18	6·6	9024·27	*p*	< 12·1	4	!
19	7·4	9025·30	*q*	< 12·6	3	!
20	6·2	9026·26	*t*2N	< 13·3	4	!
25	6·8	9031·28	*p*6N	< 12·1	4	!
26	10·0	9032·42	*k*2	11·0	3	:
27	7·6	9033·32	2*f*	10·4	3	!?
28	6·7	9034·28	=*g* ; =*k*	10·80	3	!
29	6·2	9035·26	*f*4 ; 3*g* ; 3*k*	10·67	2	!
30	7·2	9036·30	*f*3 ; 3*g* ; 3*k*	10·66	3	!
31	7·1	9037·30	*f*3 ; 3*k*	10·67	4	!
1911						
Jan. 2	13·5	9039·56	*k*6 ; =*m*	11·16	2	i
3	12·8	9040·53	=*m*	11·20	3	!
4	6·4	9041·27	*o*2	11·61	2	!
5	14·5	9042·60	=*q*	12·46	2	!
7	13·3	9044·55	*q*10N (*t*)	< 13·1	4	!
9	6·8	9046·28	*q*6N	< 12·7	3	!
17	6·6	9054·27	*p*, *q*	< 12·3	4	!
Feb. 4	7·0	9072·29	*o*, *p*	< 11·7	4	(!
18	7·0	9086·29	*p*4N	< 12·0	4	!
21	13·2	9089·55	*q*, *s*	< 12·8	4	!
22	17·7	9090·74	*q*	< 12·4	4	!
25	13·6	9093·57	*q*2N	< 12·5	3	!
26	13·5	9094·56	*p*, *q*	< 12·4	3	!
27	8·8	9095·37	*q*4N	< 12·6	4	!
28	10·0	9096·42	*q*4N	< 12·6	3	!
Mar. 1	13·8	9097·57	*q*4N	< 12·6	3	!
2	12·8	9098·53	*q*2N (*t*)	< 12·8	4	!
4	13·5	9100·56	*q*2N	< 12·5	3	!
5	13·5	9101·56	*q*0N	< 12·45	3	!
6	12·9	9102·54	*p*, *q*	< 12·4	4	!
10	13·5	9106·56	*l*	< 11·1	3	(:
15	8·5	9111·35	*p*	< 11·5	4	(!
16	8·2	9112·34	*p*, *q*	< 12·3	4	!
17	13·2	9113·55	*n* (*p*)	< 11·6	3	(!
20	7·8	9116·32	*q*2N	< 12·55	3	!
21	14·3	9117·60	*l*	< 11·0	2	;
22	7·5	9118·31	*q*2N	< 12·55	4	!
23	8·7	9119·36	*q*4N	< 12·6	4	!
29	8·1	9125·34	*p*2N	< 11·9	3	:
31	7·7	9127·32	*q*4N	< 12·6	4	!

TABLE II—*continued*

		G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1911		h					
Apr.	1	9·6	9128·40	*q*4N	< 12·6	*4*	!
	2	9·6	9129·40	*q*4N	< 12·6	*4*	!
	3	9·4	9130·39	*q*oN	< 12·45	*2*	!
	4	8·4	9131·35	*p*oN	< 11·8	*3*	(!
	8	9·7	9135·40	*k*4N	< 11·0	*3*	(!
	10	8·0	9137·33	*l*2N	< 11·1	*4*	(!
	12	8·7	9139·36	*k*6N (*f*6N)	< 11·0	*4*	(!
	13	10·2	9140·42	*k*oN	< 10·8	*3*	(!
	14	9·7	9141·40	*l*6N, *n*oN	< 11·5	*3*	(!
	15	9·8	9142·41	*q*3N	< 12·6	*4*	!
	17	9·9	9144·41	*q*oN	< 12·45	*4*	!
	19	8·0	9146·33	*q*oN	< 12·45	*3*	!
	20	8·7	9147·36	*p*	< 12·0	*3*	!
	21	10·0	9148·42	*q*2N	< 12·55	*3*	!
	22	8·3	9149·35	*q*oN	< 12·45	*3*	!
	23	8·6	9150·36	*q*	< 12·3	*3*	!
	24	8·9	9151·37	*q*4N	< 12·6	*4*	!
	27	9·8	9154·41	*p*	< 12·0	*2*	:
	29	8·7	9156·36	*p*	< 12·0	*3*	!
	30	8·6	9157·36	*p*	< 12·2	*3*	!
May	1	8·7	9158·36	*p*	< 12·2	*3*	!
	3	9·6	9160·40	*p*	< 12·0	*3*	(!
	5	9·6	9162·40	*l*, *m*	< 11·3	*3*	(!
	6	10·0	9163·42	*l*	< 11·0	*3*	(!
	7	8·9	9164·37	*l*	< 11·0	*3*	(!
	8	9·8	9165·41	*k*oN	< 10·8	*3*	(:
	13	8·9	9170·37	*k*4N	< 11·0	*3*	(:
	17	9·2	9174·38	*f*6 ; 3*k*	10·74	*3*	:
	21	9·0	9178·37	*f*4 ; 2*k*	10·72	*3*	:
	22	9·2	9179·38	=*g* ; =*k*	10·80	*3*	!
	23	10·4	9180·43	*g*4 ; *k*4	11·00	*1*	;
	25	9·8	9182·41	1*n*	11·36	*2*	:
	29	9·0	9186·37	*k*oN	< 10·8	*3*	:
July	23	14·7	9141·61	*o*	< 11·5	*2*	:
	29	15·2	9147·63	*p*6N (*r*)	< 12·2	*4*	!
	30	15·0	9148·62	*p*6N	< 12·1	*3*	!
	31	15·2	9149·63	*p*, *q*	< 12·4	*4*	!
Aug.	1	14·3	9150·60	*p*8N	< 12·1	*4*	!
	2	14·0	9151·58	*p*10	< 12·2	*4*	!
	3	15·0	9152·62	*q*2N	< 12·55	*4*	!
	4	14·0	9153·58	*p*6N	< 12·1	*3*	i
	5	13·8	9154·57	*p*4N	< 12·0	*2*	i
	6	14·3	9155·60	*p*	< 12·0	*3*	!
	7	15·7	9156·65	*p*2N	< 11·9	*3*	!t
	8	14·6	9157·61	*k*3N	< 11·1	*3*	(:
	10	15·4	9159·64	*m*, *o*	< 11·5	*3*	(:
	11	14·8	9160·62	*o*, *p*	< 11·7	*3*	(!

TABLE II—*continued*

	G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1911	h					
Aug. 12	14·2	9161·59	*k*2N	< 10·9	2	(:
13	14·5	9162·60	*m*2N	< 11·3	2	(!
17	14·2	9166·59	*k*4 ; 4*m*	11·01	3	(:
19	13·7	9168·57	=*k*	10·82	3	!
20	13·6	9169·57	=*k*	10·82	3	:
21	13·1	9170·55	2*k*	10·72	3	:
22	14·7	9171·61	*g*2 ; 2*k*	10·90	3	!
23	13·8	9172·57	=*m*	11·20	3	!
24	15·3	9173·64	=*p*	11·78	3	!
25	15·5	9174·65	*p*8 ; 2*q*	12·27	3	!
26	14·2	9175·59	*q*10 ; 2*t*	13·04	3	!
27	15·3	9176·64	*u*4	14·0	2	!
Sept. 24	15·0	9184·62	*u*6N	< 14·1	4	!
28	13·1	9308·55	*p*6N	< 12·1	4	!
Oct. 19	14·7	9329·61	*q*0N	< 12·45	4	!
24	13·7	9334·57	*q*0N	< 12·45	3	!
26	12·0	9336·50	*q*2N	< 12·55	3	!
27	12·2	9337·51	*p*8N (*q*)	< 12·4	4	:
30	12·6	9340·52	*q*2N	< 12·55	4	!
31	12·3	9341·51	*p*8N	< 12·2	4	:
Nov. 2	13·0	9343·54	*p*4N	< 12·0	3	(!
5	13·0	9346·54	*p*2N	< 11·9	3	(:
12	8·1	9353·34	*q*0N	< 12·45	4	!
14	13·7	9355·57	*p*6N	< 12·1	4	;
16	6·2	9357·26	*p*0N	< 11·8	3	!
18	12·6	9359·52	*p*2N	< 11·9	3	!
21	12·0	9362·50	*q*0N	< 12·45	3	!
22	12·7	9363·53	*p*4N	< 12·0	3	:
27	14·4	9368·60	*k*6 ; =*l* ; 2*m*	11·07	3	!
29	11·4	9370·60	*f*4 ; 4*k*	10·67	4	!
30	13·1	9371·55	*f*4 ; 4*k*	10·67	3	!
Dec. 5	12·0	9376·50	=*k*	10·82	2	(!
7	11·7	9378·53	*m*0N	< 11·2	2	(!
10	12·2	9381·51	*p*6N	< 12·1	4	(!
11	10·0	9382·42	*q*0N	< 12·45	4	!
1912						
Jan. 20	6·8	9422·28	*p*0N	< 11·8	3	!
21	7·6	9423·32	*p*6N	< 12·1	4	!
23	7·2	9425·30	*p*6N	< 12·1	4	!
24	16·0	9426·67	*p*0N	< 11·8	3	!
25	13·0	9427·54	*p*6N	< 12·1	3	!
26	12·8	9428·53	*p*4N	< 12·0	3	!
27	12·6	9429·52	*k*6N	< 11·1	3	(:
28	12·5	9430·52	*k*6N	< 11·1	4	(!
29	16·6	9431·69	*l*6N ; *m*0N	< 11·2	3	:

TABLE II—*continued*

		G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1912		h					
Feb.	4	7·0	9437·29	*p*oN	< 11·8	*3*	:
	5	8·2	9438·34	*n*, *o*	< 11·7	*3*	!
	6	14·0	9439·58	*n*6N ; o2N	< 11·7	*4*	(!
	8	7·2	9441·30	*p*6N	< 12·1	*3*	!
	9	8·0	9442·33	*q*oN	< 12·5	*3*	!
	10	7·5	9443·31	*p*8N	< 12·2	*3*	!
	11	7·1	9444·30	*p*oN	< 11·8	*3*	!
	11	12·6	9444·52	*q*oN	< 12·45	*3*	!
	13	8·7	9446·34	*p*oN	< 11·8	*3*	!
	15	8·7	9448·34	*t*4N	< 13·4	4	!
	16	7·5	9449·31	*t*4N	< 13·4	*3*	!
	17	13·0	9450·54	*o*oN	< 11·5	3	;
	18	15·6	9451·65	*p*6N	< 12·1	*3*	!
	19	12·5	9452·52	*k*4 ; *l*2 ; 6*m*	11·01	4	!
	20	7·6	9453·32	= *k*	10·82	*2*	:
	20	13·5	9453·55	*g*2 ; = *k*	10·85	*3*	:
	21	11·7	9454·53	*g*2 ; = *k*	10·85	*4*	:
	23	12·4	9456·52	*m*′2	11·30	*2*	:
	25	14·1	9458·58	*p*2 ; 8*q*	11·96	*3*	:
	26	15·0	9459·62	*q*2	12·56	*3*	
Mar.	14	8·2	9476·34	4N	< 13·2	4	!
	20	11·2	9482·47	*p*4N	< 12·0	*3*	!
	23	12·3	9485·51	*q*8N	< 12·9	3	:
	24	12·8	9486·53	*q*6N (*s*)	< 12·9	4	:
Apr.	3	8·3	9496·35	*p*oN	< 11·8	*3*	!
	4	7·8	9497·32	*p*oN	< 11·8	*3*	!
	5	11·1	9498·46	*m*oN	< 11·2	*3*	(:
	6	8·1	9499·34	*s*2N	< 13·1	4	!
	7	8·6	9500·36	*p*oN	< 11·8	*3*	!
	8	8·2	9501·34	*p*4N (*q*)	< 12·2	*4*	!
	9	8·2	9502·34	*l*2 ; 1*m*	11·10	*3*	!
	10	8·2	9503·34	*g*2 ; = *l*	10·94	*3*	!
	13	9·0	9505·37	*g*6 ; *l*2	11·03	*2*	:
	14	9·2	9507·38	= *m*	11·20	*2*	!
	15	8·7	9508·36	= *l*	11·00	*2*	!
	17	9·0	9510·37	*m*″4 ; 6*o*″	11·30	*4*	!
	18	8·1	9511·34	=o ; 4*p*	11·55	*4*	:
	19	9·7	9512·40	2*q*	12·36	*3*	:
July	31	15·3	9615·64	*l*4N ; *m*oN	< 11·2	3	(!
Aug.	8	15·7	9623·65	*q*oN (*ss* seen??)	< 12·45	3	(t!
	9	15·0	9624·62	*l*4N ; *m*oN	< 11·2	*3*	:
	10	15·0	9625·62	*s*oN	< 13·0	3	!
	11	15·7	9626·65	*t*4N	< 13·4	3	!
	12	14·5	9627·60	*l*oN	< 11·0	*3*	:
	18	14·0	9633·58	*l*2N	< 11·1	*3*	:
	20	13·3	9635·55	*q*oN	< 12·45	2	:

TABLE II—*continued*

1912	G.M.A.T. h	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
Aug. 21	15·8	9636·66	*t*2N	< 13·3	3	!
22	15·4	9637·64	*t*2N	< 13·3	3	!
23	15·4	9638·64	*t*4N	< 13·4	3	!
24	15·7	9639·65	*t*4N ; *u*oN	< 13·8	3	!
26	15·0	9641·62	*p*oN	< 11·8	3	(!
27	15·0	9642·62	*p*oN	< 11·8	*4*	(:
29	13·7	9644·57	*m*2N	< 11·3	*3*	(:
30	15·5			< 11·8		
Sept. 2	12·8	9648·53	*l*oN	< 11·0	*3*	(;
4	12·7	9650·53	*p*oN	< 11·8	*3*	(!
5	15·0	9651·62	*p*6N	< 12·1	*4*	(!
6	12·5	9652·52	*r*, *q*	< 12·4	*3*	!
7	13·3	9653·55	*q*2N	< 12·55	*3*	!
9	12·8	9655·53	*p*	< 11·8	2	:
10	14·0	9656·58	*t*oN	< 13·2	*3*	!
11	14·2	9657·59	*l*oN	< 11·0	*3*	;
13	12·8	9659·53	*p*4N	< 12·0	3	;
14	15·0	9660·62	*t*4N	< 13·4	*3*	!
15	14·0	9661·58	*o*2N	< 11·6	*3*	:
16	14·2	9662·59	*t*oN	< 13·2	4	!
17	14·6	9663·61	2*t*N	< 13·1	3	!
20	15·8	9666·66	*u*oN	< 13·8	3	!
21	14·2	9667·59	*t*oN	< 13·2	3	:
23	13·0	9669·54	*o*oN	< 11·5	3	(:
24	14·3	9670·60	*p*6N	< 12·1	3	(!
25	14·0	9671·58	*o*oN	< 11·5	?	(:
27	14·0	9673·58	*n*oN	< 11·45	3	(:
28	14·3	9674·60	*g*4 ; *k*4 ; *l*4	11·07	3	(:
Oct. 2	14·8	9678·62	*f*6 ; 3*g* ; 2*k*	10·72	3	(:
3	14·7	9679·61	*f*4 ; 2*g* ; 2*k*	10·70	3	(:
4	14·0	9680·58	*f*6 ; *g*2 ; =*k* ; 6*n*	10·93	3	!
5	13·0	9681·54	*f*6 ; =*g* ; =*k*	10·80	4	!
7	12·3	9683·51	*o*2 ; 3*p*	11·62	2	i
8	14·2	9684·59	*n*6 ; *o*2 ; 2*p*	11·68	3	!
9	13·8	9685·57	*p*8 ; 3*q*	12·25	3	!
10	14·1	9686·59	*q*6	12·76	2	:
12	14·5	9688·60	=*u* (glimpsed)	13·8	1	!
31	9·1	9707·38	*q*8N ; *t*oN	< 13·2	3	!
Nov. 1	8·0	9708·33	*p*6N	< 12·1	3	:
3	13·4	9710·56	*t*oN	< 13·2	3	!
4	13·3	9711·55	*t*oN	< 13·2	3	!
7	13·0	9714·54	*t*4N	< 13·4	3	!
8	13·3	9715·55	*q*	< 12·2	*3*	!
9	13·3	9716·55	*t*oN	< 13·2	3	!
10	9·9	9717·41	*p*oN	< 11·8	*3*	!
12	9·4	9719·39	*p*4N	< 12·0	*3*	!
13	9·5	9720·40	*t*oN	< 13·2	3	!

TABLE II—*continued*

	G.M.A.T.	J.D. 2410000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1912	h					
Nov. 14	9·7	9721·40	*p*4N	< 12·0	3	:
16	13·6	9723·57	*o*oN	< 11·5	*3*	:
18	14·5	9725·60	*m*4 ; 4*o*	11·35	*2*	!
19	13·2	9726·55	*l*6 ; *m*2 ; 6*o*	11·27	4	(!
20	9·2	9727·38	*o*2 ; 3*p*	11·62	2	(:
22	15·0	9729·62	*p*2N	< 11·9	3	(!
Dec. 14	16·7	9751·70	*t*oN	< 13·2	3	!
27	?	9764·—	*q*6N	< 12·7	3	!
1913						
Jan. 1	6·9	9769·29	*o*oN	< 11·5	*3*	!
4	6·9	9772·29	*o*oN	< 11·5	*3*	!
7	6·8	9775·28	*m*oN	< 11·2	*3*	;
8	13·0	9776·54	*m*oN	< 11·2	*3*	!
10	6·6	9778·27	*l*oN	< 11·0	*3*	:
11	9·7	9779·40	*m*oN	< 11·2	*3*	:
12	7·3	9780·30	*q*oN	< 12·45	—	(!
17	13·4	9785·56	*l*oN	< 11·0	*2*	(!
18	17·2	9786·72	*t*oN	< 13·0	3	:
25	6·7	9793·28	=*m*	11·20	3	!
27	7·0	9795·29	*q*2 ; 4*s*	12·68	3	!
Apr. 14	9·8	9872·41	*l*4N	< 11·2	?	(!
		2410000 +				
		2420000 +				
May 1	8·7	9889·37	*o*oN	< 11·5	*2*	!
7	10·6	9895·44	*n*oN	< 11·45	*2*	:
8	9·6	9896·40	*o*oN	< 11·5	*3*	:
10	10·0	9898·42	*p*6N	< 12·1	3	(:
July 12	15·2	9961·63	*p*oN	< 11·8	3	!t
16	15·4	9965·64	*p*oN	< 11·8	3	:
27	14·3	9976·60	*p*oN	< 11·8	3	(!
29	14·5	9978·60	*l*oN	< 11·0	*3*	(!
30	14·5	9979·60	*o*oN	< 11·5	*2*	:
Aug. 1	14·0	9931·58	*m*oN	< 11·2	*3*	:
2	15·2	9982·63	*p*oN	< 11·8	*3*	:
3	14·5	9983·60	*p*oN	< 11·8	*3*	:
4	14·7	9984·61	*p*4N	< 12·0	*3*	!
5	15·0	9985·62	*m*oN	< 11·2	*2*	:
6	15·5	9986·65	*o*oN	< 11·5	*3*	:
8	15·4	9988·64	*p*4N	< 12·0	*3*	!
9	15·0	9989·62	*p*6N	< 12·1	3	!
10	15·4	9990·64	*p*2N	< 11·9	3	:
11	15·3	9991·64	*o*oN	< 11·5	*3*	:
14	15·9	9994·66	*o*oN	< 11·5	3	!t
15	15·8	9995·66	*m*oN	< 11·2	*3*	(!
19	15·0	9999·62	*p*4N	< 12·0	3	(!

TABLE II—*continued*

	G.M.A.T.	J.D. 2420000 +	Comparisons or Comparison Stars Used	Mag. McC.	W.	Sky, etc.
1913	h					
Aug. 26	15·7	0006·65	*m*oN	< 11·2	*3*	(:
27	13·3	0007·55	*q*oN	< 12·45	3	(!
28	14·4	0008·60	*p*oN	< 11·8	*2*	!
29	14·0	0009·58	*t*2N	< 13·3	3	!
31	13·6	0011·57	*m*4 ; 2*o*	11·40	*2*	!
Sept. 1	15·0	0012·62	*f*6 ; 4*k*	10·72	3	!
2	15·2	0013·64	*f*6 ; =*k* ; 6*m*	10·85	3	!
4	14·7	0015·61	=*p*	11·78	*3*	!
5	14·8	0016·62	*p*6 ; 4*q*	12·17	4	!
Oct. 29	13·8	0070·57	*k*2 ; =*g*	10·85	*3*	!
31	13·5	0072·56	*k*4 ; *g*4	11·00	*2*	!
Nov. 2	15·0	0074·62	*o*6 ; 4*p*	11·70	*3*	!
1914						
Jan. 25	8·0	0158·33	2*p*N	< 11·7	*3*	!
26	7·7	0159·32	*p*oN	< 11·8	*3*	!
27	8·2	0160·34	*n*oN	< 11·45	*2*	:
28	14·0	0161·58	*q*oN	< 12·45	*2*	!
30	13·2	0163·55	*p*oN	< 11·8	*2*	!
Feb. 10	7·7	0174·32	*p*oN	< 11·8	3	(!
14	10·0	0178·42	*s*oN	< 13·0	3	:
22	9·4	0186·39	*o*oN	< 11·5	*3*	!
Mar. 18	10·5	0210·44	*m*4N	< 11·4	*2*	!
22	11·7	0214·49	*o*oN	< 11·5	*3*	!
30	9·6	0222·40	*p*oN	< 11·8	*4*	!
Apr. 6	9·2	0229·38	*l*oN	< 11·0	—	(!
20	10·2	0243·42	*l*6 ; =*m*	11·25	*2*	:
22	8·5	0245·35	=*l*	11·00	*2*	!
27	9·8	—	*l*oN	< 11·0	*3*	:
30	8·9	0253·37	*m*oN	< 11·2	*2*	!

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A MINIMUM THEOREM IN THE THEORY OF INTERNAL CONSTITUTION OF STARS.

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Professor E. A. Milne * and Dr. S. Chandrasekhar † have recently proved a series of minimum or maximum theorems of great importance concerning the internal constitution of stars. In the present paper I prove a minimum theorem the character of which is essentially different from that of the theorems proved by Milne and Chandrasekhar. The theorem I prove states that a star built up according to the equations of mechanical and radiative equilibrium can be considered as a solution of a minimum problem. A similar theorem which I have proved for Eddington's star model has been published in the *Zeitschrift für Astrophysik.*‡

The well-known equations of mechanical and radiative equilibrium can be written in the following form :—

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)+4\pi Gr^2\rho=0$$

$$\frac{d}{dr}\left(\frac{r^2}{k\rho}\frac{dp_r}{dr}\right)+\frac{4\pi}{c}\epsilon r^2\rho=0,$$

where r denotes the distance from the centre of the star, P the total pressure composed of the gas pressure p_g and the radiation pressure p_r, ρ the density, k the absorption coefficient, $4\pi\epsilon$ the amount of energy produced in one second by unit mass, G the constant of gravitation and c the velocity of light. Introducing, instead of ρ, the mass $M(r)$ contained in a sphere of radius r around the centre of the star, and instead of ϵ, its mean value $\bar{\epsilon}$ within the mentioned sphere of radius r, we can put the equations of equilibrium into the following form :—

$$\frac{d}{dM(r)}\left(r^4\frac{dP}{dM(r)}\right)+\frac{G}{4\pi}=0 \qquad (1)$$

$$\frac{d}{dM(r)}\left(\frac{r^4}{k\bar{\epsilon}}\frac{dp_r}{dM(r)}\right)+\frac{1}{4\pi c}=0, \qquad (2)$$

where

$$dM(r)=4\pi r^2\rho dr \quad \text{and} \quad \bar{\epsilon}=\frac{1}{M(r)}\int_0^{M(r)}\epsilon dM(r).$$

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† *M.N.*, **96**, 644, 1936 ; *Ap. J.*, **85**, 372, 1937 ; **86**, 78, 1937.
‡ *Zs. f. Astroph.*, **14**, 251, 1937.

To explain the theorem which I shall prove, let us consider a configuration of spherical symmetry which is not in equilibrium. In such a configuration the variables p_g, p_r, $M(r)$, k and $\bar{\epsilon}$ are arbitrary functions of the variable r; we assume that $M(r)=0$ for $r=0$, thus excluding any singularity at the centre of the configuration; naturally, the central values of p_g and p_r are positive, $M(r)$ increases with r for the density being positive in the interior of the configuration, and k and $\bar{\epsilon}$ are each positive functions of r. By varying the variables p_g, p_r, $M(r)$, k, $\bar{\epsilon}$ and r by very small amounts we obtain from the above configuration a "neighbouring configuration." Now let us choose the variations of $M(r)$, k, $\bar{\epsilon}$ and r in such a way that $M(r)$ and the product $k\bar{\epsilon}$ are, in the neighbouring configuration, the same functions of r as in the primary configuration; moreover, let us suppose that the values of p_g, p_r and r at the centre and at the boundary of the neighbouring configuration are the same as those at the corresponding places of the primary configuration. Any neighbouring configuration satisfying these conditions shall be termed "admissible."

Setting

$$I=\int_0^M [Q_1^2+Q_2^2-(a_1p_g+a_2p_r)]dM(r), \tag{3}$$

where

$$Q_1=r^2\left(\frac{dp_g}{dM(r)}+\frac{dp_r}{dM(r)}\right), \qquad Q_2=\frac{r^2}{\sqrt{k\bar{\epsilon}}}\frac{dp_r}{dM(r)},$$

$$a_1=\frac{G}{2\pi}, \qquad a_2=\frac{1}{2\pi}\left(G+\frac{1}{c}\right), \tag{4}$$

and where M is the boundary value of $M(r)$, *i.e.* the total mass of the configuration, the theorem to be proved here can be expressed as follows:—

Theorem.—A configuration which gives a minimum value to the integral I in comparison with all admissible neighbouring configurations is an equilibrium configuration (*i.e.* a configuration that satisfies the equations (1) and (2)).

Proof.—The proof of this theorem can be divided into two parts: in the first part I shall prove that only the equilibrium configuration satisfies the variation condition $\delta I=0$—in other words, for any equilibrium configuration—and for no other configuration—the value of the integral I is an extremal value, *i.e.* either a maximum or a minimum or a stationary value; in the second part of the proof I shall show that this extremal value is in fact a minimum.

1. If we form the variation of the integral I directly from (3), we cannot apply the variation to the variable $M(r)$, for, according to the laws of the calculus of variations, the integration variable must remain unvaried; thus, neither the variables r and $k\bar{\epsilon}$ can be varied, according to the condition that in admissible neighbouring configurations $M(r)$ and $k\bar{\epsilon}$ have to be the same functions of r as in the extremum configuration looked for. To overcome these restrictions let us introduce a parameter t, so that the

variables p_ϱ, p_r, $M(r)$, k, $\bar{\epsilon}$ and r will be differentiable functions of t. Introducing this parameter into (3) and setting

$$\frac{dp_\varrho}{dt} = p_\varrho', \qquad \frac{dM(r)}{dt} = M'(r), \text{ etc.},$$

we have

$$I = \int_{t_0}^{t_1} [Q_1^2 + Q_2^2 - (a_1 p_\varrho + a_2 p_r)] M'(r) dt, \tag{5}$$

where

$$Q_1 = r^2 \left(\frac{p_\varrho'}{M'(r)} + \frac{p_r'}{M'(r)} \right), \qquad Q_2 = \frac{r^2}{\sqrt{k\bar{\epsilon}}} \frac{p_r'}{M'(r)}; \tag{6}$$

here t_0 and t_1 are values of the parameter corresponding to the values 0 and M of the variable $M(r)$, and the constants a_1 and a_2 are given by (4). Let us assume that in the extremum configuration and in the admissible neighbouring configurations $M'(r)$ is positive, so that in (5) the integration is made in the direction of increasing values of t.

Now, the variables $M(r)$, r and $k\bar{\epsilon}$ can also be varied. But according to the condition put above, namely, that $M(r)$ and $k\bar{\epsilon}$ should in the neighbouring configurations remain the same functions of r as they were in the extremum configuration we are looking for, the following variation conditions should also be taken into consideration :—

$$\delta M(r) = \frac{M'(r)}{r'} \delta r, \qquad \delta(k\bar{\epsilon}) = \frac{(k\bar{\epsilon})'}{r'} \delta r. \tag{7}$$

Now, if we set

$$[Q_1^2 + Q_2^2 - (a_1 p_\varrho + a_2 p_r)] M'(r) = F, \tag{8}$$

according to the principal laws of the calculus of variations, the variation of the integral I yields :

$$\delta I = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial F}{\partial p_\varrho} - \left(\frac{\partial F}{\partial p_\varrho'} \right)' \right] \delta p_\varrho + \left[\frac{\partial F}{\partial p_r} - \left(\frac{\partial F}{\partial p_r'} \right)' \right] \delta p_r - \left(\frac{\partial F}{\partial M'(r)} \right)' \delta M(r) + \frac{\partial F}{\partial r} \delta r + \frac{\partial F}{\partial (k\bar{\epsilon})} \delta(k\bar{\epsilon}) \right\} dt. \tag{9}$$

In deriving this equation we have taken into account the fact that all variations vanish for both t_0 and t_1, because the values of the variables at the centre and at the boundary are not varied when passing on to an admissible neighbouring configuration.

Using (7) and since the variations δp_ϱ, δp_r, δr are arbitrary and independent of one another, we deduce from (9) and from the extremum condition $\delta I = 0$:

$$\left.\begin{aligned} \frac{\partial F}{\partial p_\varrho} - \left(\frac{\partial F}{\partial p_\varrho'} \right)' = 0, \qquad \frac{\partial F}{\partial p_r} - \left(\frac{\partial F}{\partial p_r'} \right)' = 0, \\ \frac{\partial F}{\partial r} - \left(\frac{\partial F}{\partial M'(r)} \right)' \frac{M'(r)}{r'} + \frac{\partial F}{\partial (k\bar{\epsilon})} \frac{(k\bar{\epsilon})'}{r'} = 0. \end{aligned}\right\} \tag{10}$$

From (8) we have

$$\left.\begin{aligned}
&\frac{\partial F}{\partial p_g} = -a_1 M'(r), \qquad \frac{\partial F}{\partial p_r} = -a_2 M'(r),\\
&\frac{\partial F}{\partial r} = \frac{4r^3}{M'(r)}\left[(p_g'+p_r')^2+\frac{p_r'^2}{k\bar{\epsilon}}\right], \qquad \frac{\partial F}{\partial (k\bar{\epsilon})} = -\frac{r^4}{(k\bar{\epsilon})^2}\frac{p_r'^2}{M'(r)},\\
&\frac{\partial F}{\partial p_g'} = \frac{2r^4}{M'(r)}(p_g'+p_r'),\\
&\frac{\partial F}{\partial p_r'} = \frac{2r^4}{M'(r)}\left(p_g'+p_r'+\frac{p_r'}{k\bar{\epsilon}}\right),\\
&\frac{\partial F}{\partial M'(r)} = -\frac{r^4}{M'(r)^2}\left[(p_g'+p_r')^2+\frac{p_r'^2}{k\bar{\epsilon}}\right]-(a_1 p_g + a_2 p_r).
\end{aligned}\right\} \tag{11}$$

Substituting this in (10) we obtain the following equations which must be satisfied by the extremum configuration :—

$$2\left[\frac{r^4}{M'(r)}(p_g'+p_r')\right]'+a_1 M'(r)=0 \tag{12}$$

$$2\left[\frac{r^4}{M'(r)}\left(p_g'+p_r'+\frac{p_r'}{k\bar{\epsilon}}\right)\right]'+a_2 M'(r)=0 \tag{13}$$

$$\frac{4r^3}{M'(r)}\left[(p_g'+p_r')^2+\frac{p_r'^2}{k\bar{\epsilon}}\right]+\frac{M'(r)}{r'}\left[\frac{r^4}{M'(r)^2}(p_g'+p_r')^2+\frac{r^4 p_r'^2}{k\bar{\epsilon}M'(r)^2}\right]'$$
$$+(a_1 p_g'+a_2 p_r')\frac{M'(r)}{r'}-\frac{r^4 p_r'^2 (k\bar{\epsilon})'}{(k\bar{\epsilon})^2 M'(r) r'}=0. \tag{14}$$

From (12) we get

$$\frac{1}{M'(r)}\left[\frac{r^4}{M'(r)}(p_g+p_r)'\right]'+\frac{a_1}{2}=0$$

or

$$\frac{d}{dM(r)}\left[r^4\frac{d(p_g+p_r)}{dM(r)}\right]+\frac{a_1}{2}=0.$$

Since p_g+p_r is equal to the total pressure P and the value of the constant a_1 is given by (4), the latter equation takes the following form :—

$$\frac{d}{dM(r)}\left(r^4\frac{dP}{dM(r)}\right)+\frac{G}{4\pi}=0\,;$$

this is identical with the equation (1) of mechanical equilibrium.

From (13) we have in an analogous way

$$\frac{d}{dM(r)}\left[r^4\frac{dP}{dM(r)}+\frac{r^4}{k\bar{\epsilon}}\frac{dp_r}{dM(r)}\right]+\frac{a_2}{2}=0.$$

Substituting here for a_2 its value got from (4), and subtracting from this equation the equation of mechanical equilibrium, we obtain the equation (2) of radiative equilibrium.

In this way we have deduced the equilibrium equations (1) and (2) from

the extremum conditions (12) and (13). Now, it is easily understood that the condition (14) does not yield a further equation. Owing to the following equation :—

$$\left[\frac{r^4}{M'(r)^2}(p_g'+p_r')^2+\frac{r^4 p_r'^2}{k\bar{\epsilon}M'(r)^2}\right]'=\left(\frac{r^4P'}{M'(r)}\right)'\frac{P'}{M'(r)}+\frac{r^4P'}{M'(r)}\left(\frac{P'}{M'(r)}\right)'$$
$$+\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'\frac{p_r'}{M'(r)}+\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\left(\frac{p_r'}{M'(r)}\right)',$$

the equation (14) can be written as follows :—

$$\frac{4r^3P'^2}{M'(r)}+\left(\frac{r^4P'}{M'(r)}\right)'\frac{P'}{r'}+\frac{r^4P'}{r'}\left(\frac{P'}{M'(r)}\right)'+a_1p_g'\frac{M'(r)}{r'}+\frac{4r^3p_r'^2}{k\bar{\epsilon}M'(r)}$$
$$+\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'\frac{p_r'}{r'}+\frac{r^4p_r'}{k\bar{\epsilon}r'}\left(\frac{p_r'}{M'(r)}\right)'+a_2p_r'\frac{M'(r)}{r'}-\frac{r^4p_r'^2(k\bar{\epsilon})'}{(k\bar{\epsilon})^2M'(r)r'}=0.$$

Taking into account that

$$\frac{4r^3P'^2}{M'(r)}+\frac{r^4P'}{r'}\left(\frac{P'}{M'(r)}\right)'=\left(\frac{r^4P'}{M'(r)}\right)'\frac{P'}{r'}$$

and

$$\frac{4r^3p_r'^2}{k\bar{\epsilon}M'(r)}+\frac{r^4p_r'}{k\bar{\epsilon}r'}\left(\frac{p_r'}{M'(r)}\right)'-\frac{r^4p_r'(k\bar{\epsilon})'}{(k\bar{\epsilon})^2M'(r)r'}=\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'\frac{p_r'}{r'},$$

we get from the above equation :

$$2\left(\frac{r^4P'}{M'(r)}\right)'\frac{P'}{r'}+a_1p_g'\frac{M'(r)}{r'}+2\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'\frac{p_r'}{r'}+a_2p_r'\frac{M'(r)}{r'}=0$$

or

$$P'\left[2\left(\frac{r^4P'}{M'(r)}\right)'+a_1M'(r)\right]+p_r'\left[2\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'+(a_2-a_1)M'(r)\right]=0$$

or

$$P'\left[\frac{1}{M'(r)}\left(\frac{r^4P'}{M'(r)}\right)'+\frac{a_1}{2}\right]+p_r'\left[\frac{1}{M'(r)}\left(\frac{r^4p_r'}{k\bar{\epsilon}M'(r)}\right)'+\frac{a_2-a_1}{2}\right]=0$$

which reduces to

$$P'\left[\frac{d}{dM(r)}\left(r^4\frac{dP}{dM(r)}\right)+\frac{G}{4\pi}\right]+p_r'\left[\frac{d}{dM(r)}\left(\frac{r^4}{k\bar{\epsilon}}\frac{dp_r}{dM(r)}\right)+\frac{1}{4\pi c}\right]=0.$$

Since, according to the equations of mechanical and radiative equilibrium which we have obtained from (12) and (13), the quantities in brackets are equal to zero, the extremum condition (14), indeed, yields no further equation.

Thus, we have obtained the result that every configuration satisfying the extremum condition $\delta I=0$, if compared with admissible neighbouring configurations, is an equilibrium configuration.

2. In the second part of the proof which follows I shall prove that the extremal value given to the integral I by the equilibrium configuration is a minimum. For this purpose I make use of Weierstrass' condition.

According to Weierstrass' theory the sign of the quantity E decides whether the value of I is a minimum or a maximum or a stationary value, the quantity E being given by the equation

$$E = F^* - p_g'^* \frac{\partial F}{\partial p_g'} - p_r'^* \frac{\partial F}{\partial p_r'} - M'(r)^* \frac{\partial F}{\partial M'(r)}; \qquad (15)$$

here the quantities $p_g'^*$, $p_r'^*$ and $M'(r)^*$ belong to a neighbouring configuration and F^* is obtained by substituting in F the quantities $p_g'^*$, $p_r'^*$, $M'(r)^*$ for p_g', p_r', $M'(r)$ respectively. Weierstrass' minimum condition which we make use of reads: If the quantity E is not negative at any point of the integration interval and if it is not equal to zero at all points of this interval, the extremal value of I is a minimum.

Using (8) and (6) we have

$$F^* = \left[r^4 \left(\frac{p_g'^*}{M'(r)^*} + \frac{p_r'^*}{M'(r)^*} \right)^2 + \frac{r^4}{k\bar{\epsilon}} \left(\frac{p_r'^*}{M'(r)^*} \right)^2 - (a_1 p_g + a_2 p_r) \right] M'(r)^*;$$

substituting this and (11) in (15) we obtain the following expression for E:

$$E = \left[r^4 \left(\frac{p_g'^*}{M'(r)^*} + \frac{p_r'^*}{M'(r)^*} \right)^2 + \frac{r^4}{k\bar{\epsilon}} \left(\frac{p_r'^*}{M'(r)^*} \right)^2 - (a_1 p_g + a_2 p_r) \right] M'(r)^* - 2r^4 (p_g' + p_r') \frac{p_g'^*}{M'(r)}$$
$$- 2r^4 \left(p_g' + p_r' + \frac{p_r'}{k\bar{\epsilon}} \right) \frac{p_r'^*}{M'(r)} + r^4 \left[(p_g' + p_r')^2 + \frac{p_r'^2}{k\bar{\epsilon}} \right] \frac{M'(r)^*}{M'(r)^2} + (a_1 p_g + a_2 p_r) M'(r)^*,$$

which can be written as follows:—

$$E = \frac{r^4 M'(r)^*}{k\bar{\epsilon}} \left[\left(\frac{p_g'^* + p_r'^*}{M'(r)^*} - \frac{p_g' + p_r'}{M'(r)} \right)^2 k\bar{\epsilon} + \left(\frac{p_r'^*}{M'(r)^*} - \frac{p_r'}{M'(r)} \right)^2 \right].$$

Hence the quantity E cannot be negative at any point because the functions $M'(r)^*$, k and $\bar{\epsilon}$ are positive, as mentioned above. And E vanishes at all points only if at all points

$$\frac{p_g'^*}{M'(r)^*} = \frac{p_g'}{M'(r)} \quad \text{and} \quad \frac{p_r'^*}{M'(r)^*} = \frac{p_r'}{M'(r)};$$

in this case we have at all points

$$\left(\frac{dp_g}{dM(r)} \right)^* = \frac{dp_g}{dM(r)} \quad \text{and} \quad \left(\frac{dp_r}{dM(r)} \right)^* = \frac{dp_r}{dM(r)};$$

hence, at the neighbouring configuration p_g and p_r are the same functions of $M(r)$ as they were at the extremum configuration, *i.e.* the neighbouring configuration is the extremum configuration itself. This case, however, can be excluded, and therefore the quantity E cannot be equal to zero at all points of the integration interval. Hence, the value of I at any equilibrium configuration is indeed a minimum if compared with all admissible neighbouring configurations.

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CALCULATION OF THE TERM IN THE MOTION OF THE LUNAR PERIGEE WITH CHARACTERISTIC e'^4.

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In Paper III of the "Stellar Problem of Three Bodies," * I showed that the first two terms of that series in powers of m which is the coefficient of e'^4 contributed 1″, 3″ respectively to the centennial motion of the lunar perigee. With the aid of certain comparisons the whole series was estimated to contribute 11″ with a margin of 2″. As this amount closed most of the gap between the calculated and observed values, it seemed worth while to calculate the series numerically. On referring back to my theory of the Motion of the Moon, I found that all the material necessary to get the new part was available and that, with the new form of the equations of motion which is being used to verify the whole theory, the calculation would not be too long. It is now completed, and the result comes out to be the same as my estimate, namely, 11″·0 with a possible error (apart from mistakes) of ± 0″·2.

The materials needed were the periodic terms with characteristics 1, e', e'^2, e'^3, e'^4, e, ee', ee'^2, ee'^3 with arguments $2iD$, $2iD \pm l$, $2iD \pm 2l'$, $2iD \pm l$, $2iD \pm l \pm l'$, $2iD \pm l \pm 2l'$, with the portions of the motion of the perigee having characteristics 1, e'^2. The terms required have the characteristic ee'^4 with arguments $2iD \pm l$.

In the new form of the homogeneous equations, the rectangular co-ordinates are referred to moving axes which follow the mean motion of the Moon instead of that of the Sun, the latter having been used by Hill. In this new form the variables are practically separated and the approximations to their solution are made by using the equations alternately instead of being obliged to solve a pair of simultaneous equations at each step. If the co-ordinates be denoted by $A(1+x)$, Ay for motion in the plane of reference, the constant A divides out and $x=0$, $y=0$ in circular motion.

Since the equations are homogeneous and of the second degree, most of the work consisted of multiplications of pairs of harmonic series, each series containing three terms; as most of the products were very small this work was done rapidly with the aid of a machine.

I conclude by giving the equations of motion in the form needed for this problem,† *i.e.* with the inclination and ratio of the parallaxes zero. With

* *M.N.*, 97, 116, 1936.

† The complete equations will appear in the *American Journal of Mathematics*, G. W. Hill memorial volume.

real variables they are if $r^2=(1+x)^2+y^2$, $D=d/ndt$.

$$\tfrac{1}{2}(1+D^2)r^2-\tfrac{1}{2}(Dx)^2-\tfrac{1}{2}(Dy)^2=3\Omega+(1+m)\frac{1}{D}\left(x\frac{\partial\Omega}{\partial y}-y\frac{\partial\Omega}{\partial x}\right)-m\frac{1}{D}\frac{\partial\Omega}{\partial l'}+\text{const.}$$

$$Dy+xDy-yDx+r^2=\frac{1}{D}\left(x\frac{\partial\Omega}{\partial y}-y\frac{\partial\Omega}{\partial x}\right)+\text{const.},$$

where Ω is the disturbing function—of the second degree in x, y—and l' is the solar mean anomaly so that $\partial\Omega/\partial l'$ vanishes with e'. In the first approximation to the solution for terms with any characteristic, all terms containing the unknowns in the first equation, except the first, can be neglected so that it gives $\frac{1}{2}\delta r^2$. The same is true for the second equation except the terms $D\delta y+\delta r^2$ and it therefore gives δy. We thence get δx and calculate the neglected parts in the two equations alternately.

This paper is an addition to Chapter VII, Section (ii) of "The Theory of the Motion of the Moon" (*Mem. R.A.S.*, **57**, 59, 1905).

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